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SPINOR FORMULATION OF MAGNETOGAS DYNAMICS

BY
LAWRENCE A. SCHMID

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GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

ABSTRACT

It is shown that all the variables needed for a classical description of the dynamical behavior of a fluid consisting of electrically charged particles having spin can be incorporated into two spinors having a total of four complex elements. The particle spin is included, not because it plays any significant role in magnetogas-dynamical problems, but because it is needed to account for all the degrees of freedom of the spinors.

The link between the spinors and the familiar quantities that describe the fluid is provided by the "particle tetrapod" consisting of one time-like and three space-like 4-vectors. These four vectors constitute an orthonormal system and are normalized to the particle density of the fluid in its local rest-frame. The time-like vector is identified with the flux density of the fluid; one space-like vector is used to specify the orientation of the particle spin axis; and the remaining degree of freedom, an angle, is postulated to be proportional to the de Broglie phase, the proportionality constant being Planck's constant.

Incorporating the de Broglie phase into the tetrapod has two consequences: First, Planck's constant is introduced into the formalism. Second, because the canonical momentum is the

gradient of the de Broglie phase, the tetrapod contains all the information needed for a dynamical, as well as a kinematical, description of the fluid.

It is shown that the vectors of the tetrapod can be generated by four different bilinear forms involving the elements of two spinors and their complex conjugates. In this sense, the spinors may be regarded as the "square roots" of the tetrapod. The phase angle common to the two spinors is the only one of the eight degrees of freedom of the spinors that does not make itself felt in generating the tetrapod. This phase angle is used to specify the sign of the particle charge.

If the spinors that describe the fluid satisfy a first-order linear partial differential equation that involves the 4-vector electromagnetic potential and the particle mass, regarded as a linear function of the gravitational potential and the specific enthalpy, then it turns out that, for the case of adiabatic flow, the quantities involved in the tetrapod automatically satisfy a 4-vector equation that has the form of the relativistic Euler equation in the presence of electromagnetic, gravitational, and pressure fields. Spin-dependent forces proportional to Planck's constant also appear in this equation but, for problems on an macroscopic scale, these forces are completely negligible. It is shown that it is possible to drop the condition of adiabatic flow, and admit fluid viscosity, or energy injection or loss (e.g. through nuclear reactions or radiation, respectively), without changing the form of the spinor equation to be solved. Only the form of the equation that determines the specific enthalpy in terms of other fluid quantities is altered.

Because the spinor equation is linear for the case of gravitational, electromagnetic, and thermal fields regarded as fixed functions of the space-time coordinates, whereas the corresponding Euler equation is nonlinear, it is suggested that use of the spinor alternative to the Euler equation would facilitate the solution of the complete magnetogas-dynamical problem, involving both fluid-dynamical and field equations, by means of a straight-forward iteration procedure. This approach should be especially fruitful in the case of the self-excited dynamo problem, and it is in terms of this problem that the physical interpretation of the formalism is made.

The spinor alternative to Euler's equation has the form of the Dirac equation, except that the particle mass is regarded as a scalar function of the space-time coordinates, rather than a constant. The theory, however, except for the incorporation of the de Broglie phase into the tetrapod, is completely classical in spirit. In particular, no quantization process, and nothing corresponding to the Exclusion Principle have been introduced into the formalism. Thus the theory developed here could not be expected to yield valid solutions for problems in which the charged fluid should become degenerate as, for instance, at the centers of certain stars.

As a preliminary to the introduction of the particle tetrapod and the spinor equation, consideration is given to the relativistic problem of electron and proton gases interacting with the gravitational, electromagnetic, and pressure fields produced by these same charged gases. The analysis is first carried out without particle spin, and is then modified to take spin effects into account. Finally, the particle tetrapod and the spinor equation are introduced. The detailed mathematical work involving spinors is relegated to an appendix, only the results being summarized and interpreted in the body of the paper, an understanding of which requires no prior knowledge of spinor analysis.

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I INTRODUCTION

Self-Excited Dynamo

The basic equations of magnetogas dynamics are composed of two categories of equations: the fluid-dynamical equations, and the field equations. It is the purpose of this paper to develop a new formulation of the fluid-dynamical equations that will simplify the problem of the simultaneous solution of the fluid-dynamical and field equations. Although this new formulation will be applicable to a wide range of problems, the problem we shall use as a guide in discussing physical questions will be that of an isolated spheriod of fully ionized plasma whose fluid and electric currents are such that it constitutes a self-excited dynamo. A discussion of this type problem and review of the literature has been given by Elsasser(1), and in a more condensed form by Cowling(2).

The only serious attempt at a quantitative solution of this problem has been the one carried out by Bullard and his collaborators(3). In this calculation the fluid dynamical half of the problem was neglected, except for imposing the continuity condition. The calculation assumes an incompressible fluid (core of the earth), and so would not be valid, except in a qualitative way, for a star. Because of the immense difficulty of the calculation, it was necessary to truncate the harmonic expansion of the solution after the first few terms, and for this reason the solution is only approximate and some doubt remains concerning its convergence. Finally, the calculation was carried out for only one possible flow pattern. Other patterns, possibly of great physical interest, have not as yet been attempted. Calculations [(4) and (5)] have been made, however, for two other flow patterns that do not pretend to resemble the actual flow within the earth's core. The primary purpose of these calculations was merely to demonstrate the possibility of the existence of a self-excited dynamo in a homogeneous, dissipative, conducting sphere. Accordingly, the flow patterns were chosen to facilitate the calculation rather than (as in Bullard's calculation) to resemble a possible flow pattern in the core of the earth. These calculations, like Bullard's, neglected the dynamical half of the problem except for the condition that the fluid flow be solenoidal.

The extreme difficulty of these calculations clearly illustrates the desirability of finding an alternative formulation of magnetogas dynamics that will ease the calculational burden.

Two-Fluid Formulation of the Problem

Rather than formulate the problem in terms of a single conducting fluid through which an electric current flows, we shall work with two electrically charged fluids - the electron and proton gases. (Simply by changing the particle mass, we would have a gas of positive ions, instead of a proton gas.) This two-fluid approach is more rigorous than the single-fluid approach in that it obviates the need to introduce the conductivity which under certain conditions becomes an untenable concept.

The electron and proton gases are postulated to be independent perfect gases that interact only through common gravitational and electromagnetic fields. The partial pressures of the two charged gases are taken into account, but viscosity is neglected. Each of the charged fluids must independently satisfy Euler's equation, which is just the expression of Newton's second law for the fluid.

Sources of Nonlinearity

The magnetogas-dynamical problem is intrinsically nonlinear inasmuch as the interaction of the electron and proton gases with each other, and the self-interaction of each, means that superposability of solutions is not possible. However, even when we eliminate the interaction by requiring that the gravitational, electromagnetic, and pressure fields be fixed functions of the space-time coordinates (i.e., independent of the fluid flow solutions), we find that Euler's equation is still nonlinear because it contains terms that are quadratic in the fluid velocity. Because we know from physical reasoning that in the case of fixed fields superposability of solutions is possible, it would appear that the nonlinearity in Euler's equation caused by the terms that are quadratic in velocity must be spurious in the sense that there must exist an alternative way to formulate the fluid-dynamical problem that would be linear for the case of fixed fields.

We shall see that it is possible to replace Euler's equation by an equivalent spinor equation that is just the desired linear alternative (in the case of fixed fields). Having this linear alternative to Euler's equation, a straight-forward iteration solution of the total magnetogas-dynamical problem is now possible: We start with a zero-order solution of the linear spinor equations (one for each of the two charged fluids), for a certain choice of zero-order fields; we use this solution to determine the zero-order source terms in the field equations; we solve the field equations for the first-order corrections to the fields, which we then use in the spinor equations to calculate the first-order corrections to the fluid flow of the electron and proton gases; then the whole process is repeated as many times as needed to give the desired accuracy. Such a straight-forward iteration procedure would be impossible using Euler's equation instead of the spinor equation because the solution of the fluid flow for the case of given fixed fields would be blocked by the nonlinearity of Euler's equation arising from the terms that are quadratic in velocity.

Spinor Equations of Motion

As a prerequisite for deriving the spinor equations, we must introduce the electron and proton spin as additional degrees of freedom. Since particle spin plays no significant role in magnetogas-dynamical problems, it would seem that this is an unnecessary complication of an already too complicated problem. In actual fact, however, in the solution of a problem like the self-excited dynamo, the introduction of spin coordinates does not increase the calculational difficulty because it merely means that we work with an expansion in spinor harmonics instead of an expansion of vector harmonics.

We shall see that it is possible to regard the spinors that describe the electron gas, for example, as the "square roots" of the electron flux density 4-vector. In the same way that we find that taking the square root of a real number gives us an extra degree of freedom (the sign of the root) that must be specified by some physical condition, taking the "square root" of the flux 4-vector gives rise to extra degrees of freedom, among which are those corresponding to particle spin. Thus, the introduction of particle spin is the necessary price we must pay for the convenience of working with the linear spinor equation, rather than with the nonlinear

Euler (4-vector) equation.

We shall find that the spinor equations of motion have the form of the Dirac equation in which the particle mass, instead of being a constant, is a scalar function of the space-time coordinates that includes the mass per particle associated with the gravitational and thermal energy of the fluid, as well as the constant rest-mass of the particle. In spite of the fact that the spinor equation of motion has the form of the Dirac equation, the theory developed in this paper is purely classical (except for the fact that it incorporates the de Broglie Hypothesis, which introduces Planck's constant into the theory). In particular, no quantization process is introduced. Thus the Exclusion Principle, and consequently degeneracy of the electron and proton gases at high densities, do not follow from the theory.

In recent years there has been an increased interest in the problem of finding a classical interpretation of the Dirac equation. Noteworthy in this respect is the long paper by Takabayasi(6), and the book by Halbwachs(7) in which most of the work up to 1960 is reviewed. More recent work has been done by Schiller(8) and by Grossmann and Peres(9). (The latter reference gives a bibliography for the most recent work.) All this work differs from the theory developed in this paper in that the authors cited were ultimately concerned with finding a deeper or more intuitively appealing interpretation of quantum mechanics, rather than in the simplification of the equations of classical magnetogas dynamics. Thus, although some formal similarities exist between the present work and certain of the papers cited, vital differences also exist, with the result that any comparison must be made with great care. In any case, the present paper is intended to be self-contained and no use is made of any previous work.

Assumptions and Approximations

A realistic solution of the self-excited dynamo problems corresponding to the sun or the core of the earth would involve taking into account many detailed and complicated physical effects, some of which are not well understood. Such a detailed program would be premature. Rather, we shall aim at solving a well-defined, relatively simple idealization that, with only minor modifications can be brought into close enough correspondence with the real problems occurring in nature to afford some physical insight. We shall now discuss the assumptions and approximations that define this idealization.

First, we treat the electron and proton gases as fluids, rather than as distributions of particles having different velocities, spins, etc. Thus, for example, at a given point of space-time, there is only a single electron velocity, and this velocity is a continuous function of the space-time coordinates.

Further, we postulate that the electron and proton gases are perfect, classical (no degeneracy) gases.

We postulate that we have at every point of space-time a fully ionized plasma in which the electrons and protons (or positive ions) are individually conserved.

We shall neglect fluid viscosity (but not pressure). This is a valid approximation in situations such as we find in the interior of the sun and earth's core, since in such cases the magnetic forces are very large compared with the viscous forces. In the case of very tenuous plasmas, such as we find in the corona of the sun, or in interstellar space, this is no longer true, and it would be necessary to take viscosity into account. This can be done in fact, by means of a relatively minor modification of the theory, which is discussed in Appendix A. For the sake of definiteness, however, we assume throughout the body of the paper that fluid viscosity may be neglected.

We impose two different adiabatic conditions: first, we postulate that a given bubble of electron or proton gas loses or gains no heat energy. Second, we postulate that if we were to make a small adiabatic virtual displacement of a given bubble of gas into a neighboring position, then the bubble will have the same thermodynamic properties as the fluid surrounding its new position. We shall see that for quasistatic (i.e., reversible) flow, these two adiabatic conditions can be combined into the single condition that the 4-gradient of the specific entropy vanish. This means that the specific entropy is constant throughout the gas for all time.

The adiabatic condition with respect to virtual displacement of a bubble of gas is a natural one, since it corresponds to the fact that even though the macroscopic flow is well-ordered, there is always small-scale turbulence that tends to maintain the assumed adiabatic condition.

The assumption that a given bubble of gas gains or loses no heat energy as time goes by is less realistic inasmuch as it neglects energy lost through radiation and the possibility of energy injection through nuclear reactions or some other mechanism. Moreover, we are neglecting heat gained or lost by conduction. It is, in fact, not difficult to modify the theory so as to allow for energy injection or loss. The way in which this can be done is discussed in Appendix A. For definiteness, however, through the body of the paper we shall neglect the possibility of energy loss or injection. We justify this by taking the point of view that we divide the total problem into two steps: (1) Solution of the dynamical problem subject to the adiabatic condition; (2) Solution of the energy balance problem subject to the assumption that dropping the adiabatic condition has no effect on the dynamical solution other than to cause a slow change in its parameters (e.g., the slow expansion or contraction of a star). This two-step point of view is illustrated in Figure 1: On the left side is shown a possible flow pattern

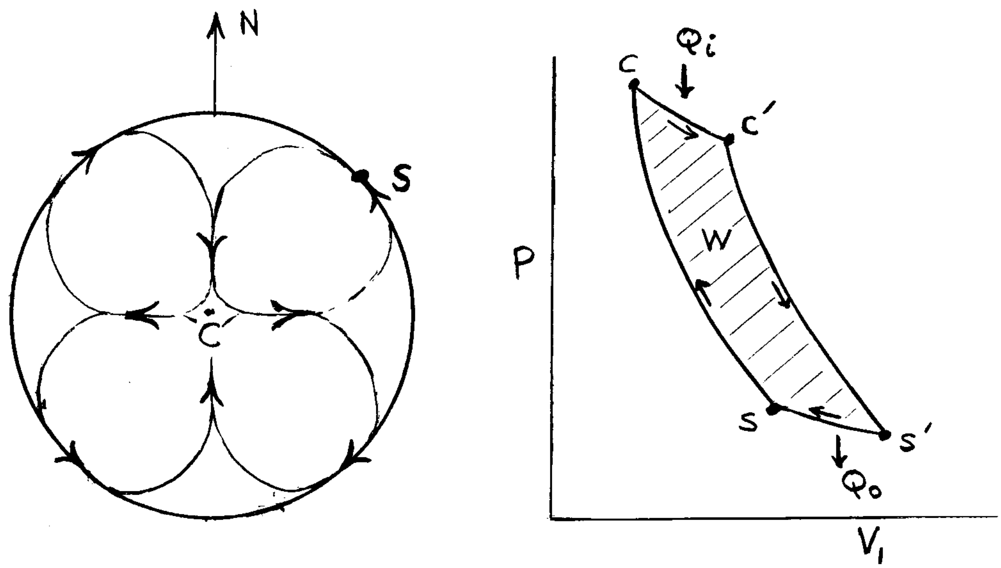

Figure 1 - Heat Transfer Associated with Convection

in the sun or the earth's core. Fluid rises from the center C in the equatorial plane, flows along the surface through a typical surface point S to the poles, where it returns to the center. On the right side of Figure 1 is shown the pressure (P) vs. specific volume (V_1) plot for a typical bubble of fluid that goes through a complete cycle in the indicated flow pattern. The points C and S on the P-V_1 plot indicate the thermodynamic state of the bubble at the corresponding points in the flow pattern. Thus, assuming the adiabatic condition, we note that as the bubble moves from the center C to the surface S, on the P-V_1 plot it moves downward from C to S, corresponding to an expansion. The return trip to the center of the core carries it along the same path in the P-V_1 plot back to the point C. Thus, on the outward trip, the bubble performs an amount of work equal to the area under the curve C-S; that is, heat energy in this amount is converted into some other form of energy, notably electromagnetic energy. But on the return trip the bubble retrieves this energy, so there is no net transfer to electromagnetic field.

Now let us imagine that, when the bubble is at C, an amount of energy Q_i is injected isothermally by means of a nuclear reaction so that while the bubble is still at the center of the core C, it traverses the isothermal C-C′. Then it expands adiabatically reaching the surface in the thermodynamic state S′. At the surface it isothermally radiates the energy Q_o, and then traverses the path S-C back to the center. Thus we have a Carnot cycle in which the net amount of energy W is transferred to the electromagnetic field to compensate losses due to Joule heating.

The two-step procedure mentioned above for solving the total problem is valid only if the Carnot cycle is very narrow. In such a case, the energy injection and loss will have no appreciable effect on the dynamics of the problem. This amounts to assuming that the energy injected and radiated during each cycle is small compared with the adiabatic work performed during the trip from the center to the surface. Obviously, this will not be the case if energy is injected too quickly. In such cases the energy injection does affect the dynamical problem, and it is necessary to modify the theory in the manner discussed in Appendix A.

As a final observation concerning the adiabatic condition, we note that, since this condition is applied to the electron and proton gases independently, it would seem that we are assuming that there is no energy transfer between the two gases; that is, we are assuming that the thermodynamic properties, in particular the temperatures, of the two gases are completely independent. We shall find, however, that it follows as a consequence of the approximate equality of the two gas densities that, if the temperatures are approximately equal at one point in space-time, then they must be approximately equal everywhere. The approximate equality of the two charged gas densities is, of course, automatically maintained by the fact that strong electric fields cannot exist in a plasma. The approximate equality of the two temperatures at a single point, such as the center of the sun or earth's core, must be imposed as a boundary condition.

Completing the list of approximations that define our idealized model, we note that we employ a scalar theory of gravitation which, although it is a covariant theory, will not give the correct higher-order velocity dependence (such as we encounter in computing the advance of the perihelion of Mercury). This requires a theory based on a symmetric tensor, as in General Relativity, rather than one based on a scalar potential.

A scalar theory, however, is perfectly adequate for the needs of magnetogas dynamics. We shall also violate the Equivalence Principle as far as the mass associated with electromagnetic, thermal, and gravitational energy is concerned. That is, although all these contributions will be included in the inertial mass, they will not be included in either the active or passive gravitational mass, for which we use just the rest-mass. This approximation should also be adequate for the needs of almost all of the existing problems in magnetogas dynamics.

Plan of Presentation

As a preliminary to writing down the fundamental equation of motion for each of the two charged fluids (Euler's equation), it is shown in Section II that the necessary thermal properties of each fluid can be completely described in terms of the specific enthalpy of the fluid and the particle density. The great advantage of doing this is that the pressure term in Euler's equation can then be replaced by the gradient of the specific enthalpy, which then plays the role of a "thermal potential function" that is completely analogous to the scalar gravitational potential. This fact allows us to absorb both the gravitational and the thermal energy into the particle rest-mass, which is regarded as a scalar function of the space-time coordinates. In this way the particle rest-mass is made to play a dual role : It describes the inertial property of the particle, and at the same time serves as a potential function for the scalar force fields acting on the particle. In Appendix A, it is shown that this procedure has the added advantage that, when we replace the adiabatic condition by one that allows for fluid viscosity or energy injection, the form of Euler's equation (or the spinor equation that replaces it) remains unchanged. Only the form of the "thermal field equation", i.e., the equation that determines the specific enthalpy as a function of the fluid variables, is altered. This same remark is also valid if we drop the assumption that the two charged fluids are perfect gases.

After writing down Euler's equation in terms of specific enthalpy instead of pressure, the rest of Section II is devoted to showing that, if we limit ourselves to a certain subset of solutions that correspond to imposing certain physically reasonable conditions, Euler's equation can be replaced by a tensor equation that is linear except for the fact that the quadratic normalization condition for the 4-velocity must be satisfied.

In Section III, it is shown that it is this tensor equation, rather than Euler's equation, that results from the Hamilton-Jacobi formulation of the problem. In order to make the discussion more physical (as well as to simplify it), the Hamilton-Jacobi formulation is presented simply as the implementation of the deBroglie Hypothesis, regarded as an experimental fact.

The conclusion drawn from Sections II and III is that the tensor equation is more fundamental than Euler's equation. In Section VIII it is shown that the spinor equations correspond to the tensor equation, rather than directly to Euler's equation. (Any solution of the tensor equation, of course, is a solution of Euler's equation, although the converse in general is not true.)

In Section IV the electromagnetic and gravitational field equations are presented.

In Section V the changes in the fluid-dynamical equations that result when we introduce particle spins are discussed.

Section VI is a summary of the results of the preceding sections, all of which have been derived within the framework of

Special Relativity, but without any mention of spinors.

In Section VII spinors are introduced as the "square roots" of 4-vectors, and all the necessary dynamical quantities are defined in terms of spinors. The discussion is intended to be self-contained, and no prior knowledge of spinor analysis is assumed.

In Section VIII the postulated spinor equation of motion is presented, and the corresponding scalar, vector, and tensor equations that follow as direct consequences of the spinor equation and the definitions of Section VII are presented and discussed. The identification of these equations with the equations of magnetogas dynamics that were summarized in Section VI is made, and the advantages of the spinor formulation are discussed. No prior knowledge of spinor analysis is necessary to follow the discussion of this section. The detailed derivation of the results presented in this section is given in Appendix C, and here some familiarity with spinor analysis is assumed.

Choice of Metric

All the work of this paper is done within the framework of Special Relativity. The metric tensor g^{jk} that is used throughout is chosen to have the following form:

$$g^{oo} = g_{oo} = 1; \; g^{jj} = g_{jj} = -1; \; g^{jk} = g_{jk} = o \qquad (j,k = 1, 2, 3) \quad j \neq k \tag{1-1}$$

This choice of metric, which is the most convenient for the transition to spinors, means that the spatial components of the contravariant and covariant forms of a 4-vector differ by a sign. We shall always identify the <u>contravariant</u> form as being the four-dimensional generalization of the corresponding 3-vector. (The only exception is the gradient operator, discussed below.) Thus, if $\vec{v}$ is the particle 3-velocity, the 4-velocity u^j has the following form:

$$u^0 = \frac{1}{\sqrt{1-\beta^2}} = u_0 ; \quad \beta = \frac{v}{c}$$

$$u^j = \frac{u^0}{c} v^j = - u_j \ (j = 1, 2, 3) \tag{1-2}$$

Note that this specification of u^j corresponds to a normalization to unity, rather than to c^2:

$$u^j u_j = 1 \tag{1-3}$$

Because (x^1, x^2, x^3) (rather than the covariant components) are identified with the coordinates (x, y, z), it follows that the <u>covariant</u> form of the gradient operator ∂_j (rather than the contravariant form ∂^j) is the generalization of the 3-gradient $\vec{\nabla} = (\partial_x, \partial_y, \partial_z)$:

$$\partial_0 = \frac{\partial}{c\partial t} = \partial^0$$

$$\partial_j = \frac{\partial}{\partial x^j} = - \partial^j \quad (j = 1, 2, 3) \tag{1-4}$$

and

$$\vec{\nabla} = (\partial_x, \partial_y, \partial_z) = (\partial_1, \partial_2, \partial_3) \tag{1-5}$$

The d'Alembertian operator $\Box$ is defined in terms of the 4-gradient ∂_j as follows:

$$\Box = \partial_j \partial^j = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \tag{1-6}$$

We shall have occasion to use the completely antisymmetric unit tensor $\varepsilon^{ijk\ell}$, which is zero if any two of its indices are equal, and +1 or -1 if the indices are all different and their order differs from the order (o 1 2 3) by an even or odd number of transpositions respectively. The covariant form of the antisymmetric unit tensor $\varepsilon_{ijk\ell}$ is derived from the contravariant form ε^{mnpq} according to the usual rule:

$$\varepsilon_{ijk\ell} = g_{im}\, g_{jn}\, g_{kp}\, g_{\ell q}\, \varepsilon^{mnpq} \tag{1-7}$$

This relation has the consequence that the definition of the covariant form of the tensor differs by a sign from that of the contravariant form. Thus $\varepsilon_{ijk\ell}$ is -1 (instead of +1) when the order of the indices $(ijk\ell)$ differs from the order (0 1 2 3) by an even number of transpositions.

Inasmuch as the definition of the contravariant form $\varepsilon^{ijk\ell}$ is just the extension of the familiar three-dimensional antisymmetric unit tensor, whereas the definition of the covariant form differs by a sign, we see that these definitions are consistent with the general rule that it is the contravariant form of a quantity, rather than the covariant form, that we regard as the generalization to four dimensions of the corresponding three-dimensional quantity.

II EULER FORMULATION

Thermodynamic Properties

As a preliminary to writing down the Euler equation for each of the charged fluids, we shall first consider the thermodynamic properties of the fluids. All these properties will appear in duplicate, one for each fluid. This fact will be indicated by the symbol $(\pm)$ used usually as a subscript, but sometimes as a superscript when the symbol in question already has a subscript. When we have a product of quantities all referring to one of the two fluids, the symbol $(\pm)$ will be appended to the entire product instead of to each symbol in the product individually. Our notation for the necessary thermodynamic quantities is defined as follows:

$P_{(\pm)}$ ≡ pressure

$T_{(\pm)}$ ≡ absolute temperature

$m_{(\pm)}$ ≡ particle rest-mass

$\rho_{(\pm)}$ ≡ particle density in fluid rest-frame

$V_1^{(\pm)}$ ≡ volume per unit mass (specific volume)

k ≡ Boltzmann's constant = 1.3708×10^{-16} erg/$^{\circ}$k

$f_{(\pm)}$ ≡ number of degrees of freedom per particle

$u_{(\pm)}$ ≡ internal thermal energy per unit mass

$\mathcal{h}_{(\pm)}$ ≡ enthalpy per unit mass (specific enthalpy)

$\mathcal{s}_{(\pm)}$ ≡ entropy per unit mass (specific entropy)

$c_v^{(\pm)}$ ≡ constant volume specific heat referred to unit mass

$c_p^{(\pm)}$ ≡ constant pressure specific heat referred to unit mass

$\gamma_{(\pm)}$ ≡ ratio of specific heat = $(c_p/c_v)_{(\pm)}$

We note that because $u_{(\pm)}$ and $\mathcal{h}_{(\pm)}$ are referred to unit mass, $(mu)_{(\pm)}$ and $(m\mathcal{h})_{(\pm)}$ represent respectively the internal thermal energy per particle, and the enthalpy per particle.

By definition

$$\mathcal{h}_{(\pm)} \equiv u_{(\pm)} + (PV_1)_{(\pm)} \qquad (2-1)$$

PV_1 is the amount of work that could be done by the gas surrounding a bubble of unit mass of the gas if this bubble were removed (or its molecules cooled to absolute zero) and the surrounding gas allowed to move in and occupy the space that had been occupied by the bubble. (We assume, of course, that the bubble is so small compared with the surrounding atmosphere of gas that, when the surrounding gas moves in to occupy the space previously occupied by the bubble, the pressure P of the gas does not change significantly. Because the energy PV_1, as well as the energy u, is theoretically available for conversion into work, h, rather than u, is to be regarded as the potential thermal energy associated with a bubble of unit mass of the gas. (Because the energy PV_1 is really provided by the gas surrounding the bubble, rather than the gas within the bubble, it would be more accurate to say that h is the potential thermal energy to be associated with the region of space occupied by the bubble.)

The above argument indicates why h is often referred to as "the heat function" and is regarded as the total thermal energy content of the gas. It is not surprising that, as we shall see below, mh, rather than mu, is the thermal potential energy of the particle, analogous to the gravitational potential energy mg, where g is the gravitational scalar potential.

Further intuitive support for the idea that it is h, rather than u, that plays the role of a thermal potential comes from the fact that, for adiabatic flow of a gas through an insulated pipe in which the gas performs no work, the governing equation has the following form: kinetic energy + potential energy + enthalpy = constant. Thus the enthalpy enters the equation in the same way as potential energy. In the case of large-scale adiabatic convection flow within a fluid, we may think of the fluid as flowing in pipes whose walls are defined by the flow lines.

The above arguments are intended only to provide an intuitive understanding of the results ensuing from the formalism below, but no use will be made of them in the formalism.

We now postulate that both charged fluids are perfect gases. Thus the following relations are valid:

$$c_v^{(\pm)} = \frac{k}{2}\left(\frac{f}{m}\right)_{(\pm)} \tag{2-2a}$$

$$c_p^{(\pm)} = c_v^{(\pm)} + \frac{k}{m_{(\pm)}} = \frac{k}{m_{(\pm)}}\left(1 + \frac{1}{2} f_{(\pm)}\right) \tag{2-2b}$$

$$\gamma_{(\pm)} = \left(c_p / c_v\right)_{(\pm)} = 1 + \frac{2}{f_{(\pm)}} \tag{2-3}$$

$$u_{(\pm)} = \left(c_v T\right)_{(\pm)} \tag{2-4}$$

$$h_{(\pm)} = \left(c_p T\right)_{(\pm)} \tag{2-5}$$

$$P_{(\pm)} = k\left(\rho T\right)_{(\pm)} \tag{2-6}$$

$$s_{(\pm)} = \left[c_v \ln T - \frac{k}{m} \ln \rho + \text{constant}\right]_{(\pm)} \tag{2-7}$$

We choose $h_{(\pm)}$ and $\rho_{(\pm)}$ as our thermodynamic variables, and note that the other quantities can be expressed in terms of these variables as follows:

$$P_{(\pm)} = k\left(\frac{\rho h}{\kappa_p}\right)_{(\pm)} = \left[\rho m h\left(1-\frac{1}{\gamma}\right)\right]_{(\pm)} \tag{2-8}$$

$$T_{(\pm)} = \left(h/\kappa_p\right)_{(\pm)} = \left[\frac{m h}{k(1+\frac{1}{2}f)}\right]_{(\pm)} \tag{2-9}$$

$$u_{(\pm)} = \left(h/\gamma\right)_{(\pm)} \tag{2-10}$$

$$s_{(\pm)} = \left[\frac{k}{m}\left(\tfrac{1}{2}f \ln h - \ln \rho\right) + \text{constant}\right]_{(\pm)} \tag{2-11}$$

From (2-1) we have

$$\partial_j h_{(\pm)} = \left[(\partial_j u + P\,\partial_j V_1) + \frac{1}{\rho m}\,\partial_j P\right]_{(\pm)} \tag{2-12}$$

where we have used the fact that, by definition

$$V_1^{(\pm)} = \frac{1}{(\rho m)_{(\pm)}} \tag{2-13}$$

Since $\Delta u + P \Delta V_1$ is the heat injected into unit mass of the gas during the change indicated by the increment symbol Δ, we can interpret $(\Delta x^j)(\partial_j u + P \partial_j V_1)$ as the heat injected into unit mass of the gas during the virtual displacement (in time or space) described by Δx^j. Thus $\partial^j u + P \partial^j V_1$ can be regarded as the 4-force per unit mass caused by heat injection. Let Q^j be this force. Thus

$$Q^j_{(\pm)} = \left[\partial^j u + P \partial^j V_1 \right]_{(\pm)} \tag{2-14}$$

We note that, using (2-8) and (2-1[illegible]), we can [illegible] (2-[illegible]) as follows:

$$\left[\rho \partial^j h - (\gamma - 1) h \partial^j \rho - \rho \gamma Q^j \right]_{(\pm)} = 0 \tag{2-15}$$

Substituting (2-14) into (2-12), we have

$$\left[\rho m \partial^j h \right]_{(\pm)} = \left[\rho m Q^j + \partial^j P \right]_{(\pm)} \tag{2-16}$$

In order to exploit this relation, we shall introduce a variable particle mass $\breve{m}_{(\pm)}$, defined as follows:

$$\breve{m}_{(\pm)} = m_{(\pm)} \left(1 + g/c^2 + h_{(\pm)}/c^2 \right) \tag{2-17}$$

where g is the gravitational potential. Thus $\breve{m}_{(\pm)}$ is the total particle mass, taking the mass associated with the gravitational and thermal energy as well as the rest-mass, into account. Because, in the situations in which we shall be primarily interested (e.g., the interior of the sun), the positive thermal energy is insufficient to overcome the negative gravitational energy, we shall refer to $\breve{m}_{(\pm)}$ as the "bound particle mass" and $m_{(\pm)}$ as the "free particle mass".

From (2-16) and (2-17), we have

$$\left[\rho\, \partial^{j}(\breve{m}c^{2})\right]_{(\pm)} = \left[\rho m\, \partial^{j} g + \partial^{j} P + \rho m\, Q^{j}\right]_{(\pm)} \qquad (2\text{-}18)$$

The significance of this relation becomes apparent when we note that the three terms on the right are the force-per-unit-volume terms that appear in Euler's equation. (Equation (2-18) has been written in contravariant form to emphasize that this, rather than the covariant form, is the generalization of 3-force.)

Euler's equation for the charged fluids has the following form:

$$\left[\rho \frac{d(\breve{m} c u^{j})}{d\tau}\right]_{(\pm)} = \left[\rho m\, \partial^{j} g + \partial^{j} P + \rho m\, Q^{j}\right]_{(\pm)} \pm q F^{jk}(\rho u_{k})_{(\pm)} \qquad (2\text{-}19)$$

where

$\tau_{(\pm)} \equiv$ proper time

$q \equiv$ magnitude of particle charge (always positive)

$F^{jk} \equiv$ Maxwell field tensor

In terms of the electric field intensity $\vec{E}$ and the magnetic flux density $\vec{B}$, the tensor F^{jk} is defined or follows:

$$\begin{aligned} (F^{10}, F^{20}, F^{30}) &\equiv (E_x, E_y, E_z) = \vec{E} \qquad \text{(statvolts/cm)} \\ (F^{23}, F^{31}, F^{12}) &\equiv -(B_x, B_y, B_z) = -\vec{B} \qquad \text{(Gauss)} \\ F^{jk} &= -F^{kj} \end{aligned} \tag{2-20}$$

where the units have been given to indicate that we shall use Gaussian units throughout. (All dynamical quantities will be expressed in cgs absolute units.)

Referring to (1-2), we note the following:

$$\frac{d}{d\tau} = c\, u^j \partial_j = u^0 \left[\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right] \tag{2-21}$$

Using (2-20) and (2-21), we can write (2-19) in the more familiar three-dimensional form:

$$\left\{ (\rho u^0) \left[\frac{\partial (\breve{m} u^0 c^2)}{\partial t} + \vec{v} \cdot \vec{\nabla} (\breve{m} u^0 c^2) \right] \right\}_{(\pm)} = \left[\rho m \frac{\partial g}{\partial t} + \frac{\partial P}{\partial t} + \rho m c Q^0 \right]_{(\pm)} \pm q (\rho u^0 \vec{v})_{(\pm)} \cdot \vec{E} \tag{2-22a}$$

$$\left\{(\rho u^0)\left[\frac{\partial(\breve{m}u^0\vec{v})}{\partial t} + \vec{v}\cdot\vec{\nabla}(\breve{m}u^0\vec{v})\right]\right\}_{(\pm)} = \left[-\rho m\vec{\nabla}g - \vec{\nabla}P + \rho m\vec{\varphi}\right]_{(\pm)} \pm q(\rho u^0)_{(\pm)}\left(\vec{E} + \frac{1}{c}\vec{v}_{(\pm)}\times\vec{B}\right) \qquad (2\text{-}22b)$$

where

$$\vec{\varphi}_{(\pm)} = (\varphi^1, \varphi^2, \varphi^3)_{(\pm)} = -(\varphi_1, \varphi_2, \varphi_3)_{(\pm)} \qquad (2\text{-}23)$$

The physical interpretation of (2-22) is immediately evident when we recall that ρu^0 and $\breve{m}u^0$ are respectively particle density and particle mass <u>in the observer's reference frame</u>. $(\rho mc\varphi^0)_{(\pm)}$ obviously represents the rate at which energy is injected per unit volume by nuclear or other reactions, and/or the heat energy generated by fluid viscosity, and $(\rho m\vec{\varphi})_{(\pm)}$ represents the viscous force, and/or the effective force resulting from heat injection caused by nuclear or other reactions.

The advantage of working with specific enthalpy becomes apparent when we substitute (2-18) into (2-19):

$$\left[\frac{d(\breve{m}cu^j)}{d\tau}\right]_{(\pm)} = \partial^j(\breve{m}_{(\pm)}c^2) \pm qF^{jk}u_k^{(\pm)} \qquad (2\text{-}24)$$

We have assumed in writing (2-24) that $\rho_{(\pm)}$ is everywhere non-zero, so that it is permissible to divide an equation through by it.

The advantage of (2-24) over (2-19) is two-fold: First, we have eliminated $Q^j_{(\pm)}$ from Euler's equation, and transferred it instead to (2-15) which may be regarded as a "thermal field equation" on a par with the gravitational and electromagnetic field equations. Thus, regardless of whether or not we introduce fluid viscosity or heat injection by nuclear reactions, we can always regard (2-24), with $\breve{m}_{(\pm)}$ defined by (2-17), as the governing fluid dynamical equation for each charged fluid. In fact, this is true even if we drop the postulate that the charged fluids are perfect gases. Second, $\rho_{(\pm)}$ does not appear in (2-24), which has the form of a single-particle equation. This would appear to allow us to solve first for $u^j_{(\pm)}$ without concerning ourselves about $\rho_{(\pm)}$ and then, knowing $u^j_{(\pm)}$, we would find $\rho_{(\pm)}$ from the continuity equation:

$$\partial_j(\rho u^j)_{(\pm)} = 0 \tag{2-25}$$

whose three-dimensional form is

$$\frac{\partial(\rho u^0)_{(\pm)}}{\partial t} + \vec{\nabla}\cdot[(\rho u^0)\vec{v}]_{(\pm)} = 0 \tag{2-26}$$

In actual fact, of course, $\rho_{(\pm)}$ enters into (2-24) implicitly through $h_{(\pm)}$ which, as we see from (2-15), is a function of $\rho_{(\pm)}$.

In an iteration solution, however, we may regard $h_{(\pm)}$ as a specified function of the space-time coordinates. In such a case, use of (2-24) allows a separation of the solution of $\rho_{(\pm)}$ and $u^{j}_{(\pm)}$ whereas (2-19) does not.

Thus far we have not introduced the assumption that our fluid flow is reversible. If we now introduce this assumption, we have the following relation:

$$Q^{j}_{(\pm)} = \left[T\, \partial^{j} s\right]_{(\pm)} = \left[\frac{h}{c_p}\, \partial^{j} s\right]_{(\pm)} \tag{2-27}$$

Using this, we can integrate (2-15) to obtain the following expression for $h_{(\pm)}$:

$$h_{(\pm)} = \left[H\left(\rho/\rho_0\right)^{(\gamma-1)} e^{(s-s_0)/c_v}\right]_{(\pm)} \qquad \text{(reversible flow)} \tag{2-28}$$

where $\rho_0^{(\pm)}$ and $s_0^{(\pm)}$ are the density and specific entropy respectively at a certain fixed point $x^{j}_{(0)}$ in space-time (and hence constants). $H_{(+)}$ and $H_{(-)}$ are two constants that must be specified as part of the boundary conditions of the problem. Obviously,

$$H_{(\pm)} = \left[h(x^{j}_{(0)})\right]_{(\pm)} = \left[c_p\, T(x^{j}_{(0)})\right]_{(\pm)} \tag{2-29}$$

Thus, the constants $H_{(+)}$ and $H_{(-)}$ can be determined from a knowledge of the fluid temperatures at the point $\overset{j}{\mathcal{N}}_{(0)}$.

The equation (2-28) may be used as the "thermal field equation" in preference to (2-15). Equation (2-15) has the advantage, however, that it is valid even when condition (2-27) is not satisfied. If, for example, we wished to introduce viscosity into the problem by requiring that $\overset{j}{\varphi}$ be a suitable function of the derivatives of $\overset{j}{u}$, then in general (2-27) will not be satisfied. In such a case (2-28) would not be valid, whereas (2-15) would.

We have indicated the way in which viscosity or heat injection through nuclear reactions would be introduced into the problem by specifying an appropriate functional form for $\overset{j}{\varphi}_{(\pm)}$. We have shown that such a procedure would in no way alter our basic dynamical equation (2-24). Only the "thermal field equation" (2-15) would be altered. The ultimate purpose of this paper is to find a spinor alternative to the vector equation (2-24), and the way we choose to handle $\overset{j}{\varphi}_{(\pm)}$ makes no difference in the form of this equation. Both for the sake of definiteness, and because we noted in Section I that the adiabatic case was a sensible first step of a two-step process for solving the total self-excited dynamo problem, we shall now introduce the <u>adiabatic condition</u>:

$$\overset{j}{Q}_{(\pm)} = 0 \qquad \textbf{(adiabatic condition)} \qquad (2\text{-}30)$$

In Appendix A several alternatives to (2-30) are discussed, but for the rest of the body of the paper we shall assume that (2-30) holds. Thus the "thermal field equation" (2-15) becomes

$$\left[\rho\,\partial_j h - (\gamma-1)\,h\,\partial_j \rho\right]_{(\pm)} = 0 \qquad \text{(adiabatic case)} \qquad (2\text{-}31)$$

and its integral is

$$h_{(\pm)} = \left[H\,(\rho/\rho_0)^{(\gamma-1)}\right]_{(\pm)} \qquad \text{(adiabatic case)} \qquad (2\text{-}32)$$

We may use either (2-31) or (2-32) as our adiabatic "thermal field equation".

Now we note that if the following relations are valid

$$\gamma_{(+)} = \gamma_{(-)} \qquad (2\text{-}33a)$$

$$\rho_{(+)} \approx \rho_{(-)} \qquad (2\text{-}33b)$$

$$(m\,H)_{(+)} \approx (m\,H)_{(-)} \qquad (2\text{-}33c)$$

then it follows from (2-32) that

$$(m\,h)_{(+)} \approx (m\,h)_{(-)} \qquad (2\text{-}34)$$

and from (2-8) and (2-9) that

$$P_{(+)} \approx P_{(-)} \tag{2-35}$$

and

$$T_{(+)} \approx T_{(-)} \tag{2-36}$$

(2-33a) is valid because the electron and proton gases are both monatomic gases. The approximate equality (2-33b) will be maintained automatically by the electrostatic screening effect. (2-33c) must be imposed as a boundary condition. From (2-2b) and (2-29) it is evident that this amounts to requiring that the electron and proton temperatures be approximately equal at a particular point in space-time (e.g., the center of the sun at t = 0). (2-35) and (2-36) then tell us that the two pressures and temperatures are approximately equal throughout space for all time. The significance of this statement is that, even through our way of formulating the problem in terms of two independent specific enthalpies $h_{(+)}$ and $h_{(-)}$ would at first glance appear to let the electron and proton pressures and temperatures differ wildly from each other, in actual fact (if we impose the boundary condition (2-34)) the two pressures and the two temperatures are automatically constrained to remain very nearly equal to each other. This is a satisfying state of affairs since we know that because of particle collisions the two temperatures cannot differ greatly.

Tensor Alternative to Euler's Equation

Equation (2-24), which is the form of Euler's equation that we shall use throughout, is a 4-vector equation. We shall now derive an alternative equation in which all the terms are antisymmetric tensors. This is most easily done by starting with the three-dimensional form given in (2-22b) (except that the bracket on the right is replaced by $-\left[\rho\vec{\nabla}(\breve{m}c^2)\right]_{(\pm)}$ and the term $\left[\vec{v}\cdot\vec{\nabla}(\breve{m}u^\circ\vec{v})\right]_{(\pm)}$ on the left is transformed by means of a well-known vector identity):

$$\frac{\partial(\breve{m}u^\circ\vec{v})_{(\pm)}}{\partial t} + \vec{\nabla}(\breve{m}u^\circ c^2)_{(\pm)} - \vec{v}_{(\pm)}\times\left[\vec{\nabla}\times(\breve{m}u^\circ\vec{v})_{(\pm)}\right] = \pm q\left(\vec{E} + \frac{1}{c}\vec{v}_{(\pm)}\times\vec{B}\right) \qquad (2\text{-}37)$$

It can be shown that (2-22a) results from the dot product of both sides of (2-37) with $\vec{v}_{(\pm)}$.

Now we note that if we impose the condition

$$\vec{\nabla}\times(\breve{m}u^\circ\vec{v})_{(\pm)} = \mp\frac{q}{c}\vec{B} \qquad (2\text{-}38)$$

(2-37) reduces to

$$\frac{\partial(\breve{m}u^\circ\vec{v})_{(\pm)}}{\partial t} + \vec{\nabla}(\breve{m}u^\circ c^2)_{(\pm)} = \pm q\vec{E} \qquad (2\text{-}39)$$

These two 3-vector equations can be combined into the following single tensor equation:

$$\partial^j(\breve{m}cu^k)_{(\pm)} - \partial^k(\breve{m}cu^j)_{(\pm)} = \mp \frac{q}{c} F^{jk} \tag{2-40}$$

This is the tensor alternative to Euler's equation (2-24). Regarding $\breve{m}$ and F^{jk} as fixed functions of the space-time coordinates, (2-40) is a linear equation, whereas Euler's equation is nonlinear. The linearity of (2-40) is spoiled, however, by the fact that we must maintain the normalization condition (1-3) for the 4-velocity:

$$(u^j u_j)_{(\pm)} = 1 \tag{2-41}$$

Thus we must still contend with the algebraic nonlinearity of (2-41). In the case of Euler's equation, the condition (2-41) must also be satisfied. In addition, we have what we might term a "differential nonlinearity" in Euler's equations itself because of the presence of the term $[u^\ell \partial_\ell(\breve{m}cu^j)]_{(\pm)}$. Thus, by replacing (2-24) with (2-40), we have rid ourselves of the differential nonlinearity, but are still left with the algebraic nonlinearity (2-41).

It is evident that the tensor equation (2-40) is not exactly equivalent to Euler's equation, because it corresponds only to the subset of solutions of (2-37) that satisfy the condition (2-38).

Either we regard (2-37) as more fundamental than (2-40), with the result that by working with (2-40) we are excluding physically meaningful, and perhaps important, solutions from consideration; or, we take the point of view that (2-38) is a physically necessary condition, and the solutions of (2-37) that do not satisfy (2-38) have no physical validity. In order to help convince ourselves that the latter situation is actually the case, we shall first investigate the physical meaning of condition (2-38).

To this end, we let $\overset{\circ}{\vec{B}} = [\vec{B}]_{v_{(\pm)}=0}$ be the magnetic field in the local fluid rest-frame.

Then from (2-38) we have

$$\left[(\vec{\nabla} \times \vec{v})_{v=0}\right]_{(\pm)} = -\frac{q}{\check{m}_{(\pm)}c}\overset{\circ}{\vec{B}} \tag{2-42}$$

The left-hand side is the fluid vorticity. The presence of vorticity means that at the microscopic level the fluid can be regarded as rotating like a rigid body with the angular velocity $\overset{\circ}{\vec{\Omega}}_{(\pm)}$ where

$$\overset{\circ}{\vec{\Omega}}_{(\pm)} = \frac{1}{2}\left[(\vec{\nabla} \times \vec{v})_{v=0}\right]_{(\pm)} \tag{2-43}$$

Thus we have

$$\overset{\circ}{\vec{\Omega}}_{(\pm)} = \mp\frac{q}{2\check{m}_{(\pm)}c}\overset{\circ}{\vec{B}} \tag{2-44}$$

which is just the Larmor Condition. The physical interpretation of this condition is that there can be no local vorticity, or rotation, in the charged fluid without a magnetic field to provide, via the Lorentz force, the necessary Coriolis force.

From (2-38) we note that large-scale vorticity ($\vec{\omega} \neq 0$) is possible even in the absence of a magnetic field, because then the gravitational force and/or the pressure gradient can provide the necessary Coriolis and centripetal forces.

Thus we see that for a charged fluid, (2-38) is a physically necessary condition. It must be emphasized that the Larmor Condition is necessary for a fluid, but not for a single particle. In the case of a single particle moving in a circular orbit, for example, the condition relating angular velocity and magnetic field is that the Lorentz force must provide the necessary centripetal force. This condition, which defines the so-called cyclotron frequency, is identical to (2-44) except that the factor 2 is absent in the denominator on the right. The Euler equation guarantees that this condition is fulfilled, since it is just the generalization of the condition on the Lorentz force for arbitrary particle orbits.

In the case of a fluid, we must allow for the fact that, in addition to the rotational velocity which requires a centripetal force, there is a random thermal velocity which, together with the rotation of the fluid, gives rise to a Coriolis acceleration which must also be taken into account. For given angular velocity, the required centripetal force is proportional to the size of the fluid vortex being considered, and so vanishes in the limit of zero vortex radius, which is the limit involved in the evaluation of $\vec{\nabla} \times \vec{v}$ in (2-43). The Coriolis force, however, remains finite in this limit since the thermal velocity on which it depends is independent of vortex size. This is the reason why, for a fluid, the Lorentz force must be equated to the Coriolis force alone.

We noted that, in the case of a single particle, the Euler equation itself guaranteed that the Lorentz force would have the proper value in relation to the characteristics of the particle orbit. In the case of a fluid, however, an extra side condition, namely (2-38), is needed to guarantee that the magnetic field strength will have the proper relation to the vorticity. The reason for the need for an extra side condition is that this condition arises from the presence of thermal velocity, which is not represented in the Euler equation. (Only the <u>energy</u> of thermal motion is represented via the enthalpy, but the detailed <u>momentum</u> and <u>force</u> considerations associated with thermal motion - namely the need for a Coriolis force - cannot be represented in Euler's equation.)

Thus, we conclude that the solutions of (2-37) which do <u>not</u> satisfy the condition (2-38), and hence are lost when we use (2-40) instead of (2-37), are those corresponding to an absolutely cold fluid with no thermal motion. In such a case there is no need to impose the Larmor Condition, and the problem of calculating fluid flow is no different from that of calculating particle orbits. If, however, thermal motions are present in the fluid (i.e., $h \neq 0$), the Larmor Condition must be imposed. Except for this extra side condition, the fluid problem is still the same as the problem of calculating particle orbits (except that we must introduce the thermal potential h), since solutions of the tensor equation (2-40) are always solutions of (2-24), which has the form of a single particle equation.

We have used above a physical argument to convince ourselves that the tensor equation is more fundamental (for real fluids) than Euler's equation. In the next section we shall use a more formal argument to arrive at the same conclusion, in that we shall show that the Hamilton-Jacobian formulation of the magnetogas-dynamical problem leads to the tensor equation, rather than to Euler's equation. To simplify the discussion, as well as to lend it greater physical significance, we shall present the Hamilton-Jacobian formulation as the incorporation of the de Broglie Hypothesis, regarded as an experimental fact, into the formalism.

III HAMILTON - JACOBI FORMULATION

We take as our starting point the following well-known expression for the canonical particle momentum p^j in the presence of an electromagnetic field represented by the 4-vector potential A^j:

$$p^j_{(\pm)} = (\breve{m} c u^j)_{(\pm)} \pm \frac{q}{c} A^j \tag{3-1}$$

The feature of the Hamilton-Jacobi formalism that is vital to our present considerations is the fact that p^j can be represented as the gradient of a scalar function Φ, which is Hamilton's Characteristic Function. Thus

$$p^j_{(\pm)} = -\partial^j \Phi_{(\pm)} \tag{3-2}$$

or, in three-dimensional form

$$p^0_{(\pm)} = -\frac{1}{c}\frac{\partial \Phi_{(\pm)}}{\partial t} \tag{3-3a}$$

$$\vec{p}_{(\pm)} = (p^1, p^2, p^3) = \vec{\nabla} \Phi_{(\pm)} \tag{3-3b}$$

From the point of view of the Hamilton-Jacobi formalism, (3-2) is part of a contact transformation that carries the Hamiltonian function over into a form that allows an easy solution. It is physically more significant, however, if we take the point of view that (3-2) is just the expression of the de Broglie Hypothesis, which we regard as an experimental fact that must be incorporated into our theory. In its usual form, the de Broglie Hypothesis states that for every particle there exists a scalar function whose gradient is the particle momentum. In order to adapt this statement to a fluid, it is necessary to impose the requirement that Φ be a continuous function of the space-time coordinates. Thus, in the same way that the fluid picture of a gas (as opposed to the kinetic theory picture) requires that the fluid velocity must not vary in a rapid and random way, but rather must be a smoothly varying function, we require that the phase function of the fluid must also be a smoothly varying function. This continuity postulate is implicit in (3-2).

Substituting (3-2) into (3-1), we have

$$-\partial^j \Phi_{(\pm)} = (\check{m} c u^j)_{(\pm)} \pm \frac{q}{c} A^j \tag{3-4}$$

Taking the curl of (3-4), we have

$$\partial^j(\check{m} c u^k)_{(\pm)} - \partial^k(\check{m} c u^j)_{(\pm)} = \mp \frac{q}{c}(\partial^j A^k - \partial^k A^j) = \mp \frac{q}{c} F^{jk} \tag{3-5}$$

where we have used the fact that the curl of A^j is the Maxwell field tensor F^{jk}. This equation is just the tensor alternative (2-40) to Euler's equation that was introduced in Section II. As was noted there, this equation (like Euler's) must be supplemented by the continuity and normalization conditions:

$$\partial_j(\rho u^j)_{(\pm)} = 0 \tag{3-6}$$

$$(u^j u_j)_{(\pm)} = 1 \tag{3-7}$$

If we solve (3-4) for $u^j_{(\pm)}$ and substitute into (3-7), we obtain

$$\left(\partial_j \Phi_{(\pm)} \pm \frac{q}{c} A_j\right)\left(\partial^j \Phi_{(\pm)} \pm \frac{q}{c} A^j\right) = (\breve{m}_{(\pm)} c)^2 \tag{3-8}$$

which is just the well-known Hamilton-Jacobi equation.

If we regard A^j and $\breve{m}_{(\pm)}$ as given space-time functions, then in principle we can solve (3-8) for $\Phi_{(\pm)}$ and use this in (3-4) to find $u^j_{(\pm)}$, which would then automatically satisfy (3-7). (It would still be necessary to impose (3-6).)

Alternatively, we could ignore (3-8), and regard (3-5) - (3-7) as our basic system of equations.

To return to the question of weighing the tensor equation (3-5) against Euler's equation (2-24), we note that if we contract (3-5) with u_j, and use the relations

$$u_j \partial^j = \frac{1}{c} \frac{d}{d\tau} \tag{3-9}$$

and

$$u_j \partial^k u^j = \frac{1}{2} \partial^k (u_j u^j) = 0 \tag{3-10}$$

the result is

$$\left[\frac{d(\breve{m} c u^k)}{d\tau}\right]_{(\pm)} = \partial^k(\breve{m} c^2)_{(\pm)} \pm q F^{kj} u_j^{(\pm)} \tag{3-11}$$

which is just Euler's equation.

Thus we have seen that it is the tensor equation, rather than Euler's equation, that results from the Hamilton-Jacobi formulation of the problem, and that Euler's equation (which is nonlinear) results from the tensor equation (which is linear) by contracting the latter with u_j. It is this contraction process that introduces the [illegible] in u_j, and hence the nonlinearity, into Euler's equation. Thus the nonlinearity of Euler's equation can be regarded as physically spurious, resulting from the unnecessary contraction of the tensor equation with u_j.

IV FIELD EQUATIONS

It was noted in Section I that the purpose of this paper is to transform the dynamical half of the magnetogas-dynamical problem, replacing Euler's equation by a spinor equation. The source equations for the electromagnetic and gravitational fields will be just Maxwell's equation and the four-dimensional Poisson equation respectively. In order to define the notation and to provide a convenient reference, the relevant field equations will be briefly discussed in this section.

Electromagnetic Field

The definition for the elements of the Maxwell field tensor F^{jk} in terms of the electric field intensity $\vec{E}$ and the magnetic flux density $\vec{B}$ has been given in (2-20). The four-dimensional form of Maxwell's equations (in unrationalized Gaussian units) is

$$\partial_j F^{jk} = 4\pi q\left[(\rho u^k)_{(+)} - (\rho u^k)_{(-)}\right] \tag{4-1}$$

where $q = 4.802 \times 10^{-10}$ statcoulomb is the magnitude (always positive) of the electron and proton charge, and $(\rho u^k)_{(+)}$ and $(\rho u^k)_{(-)}$ are the flux densities of the proton and electron gases respectively. The three-dimensional form of (4-1) is

$$\vec{\nabla}\cdot\vec{E} = 4\pi q\left[(\rho u^0)_{(+)} - (\rho u^0)_{(-)}\right] \tag{4-2a}$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c} q\left[(\rho u^{\circ}\vec{v})_{(+)} - (\rho u^{\circ}\vec{v})_{(-)}\right] + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \tag{4-2b}$$

The field tensor F^{jk} is related to the 4-vector potential in the usual way:

$$F^{jk} = \partial^j A^k - \partial^k A^j \tag{4-3}$$

The vector equivalent of (4-1) is

$$\Box A^k - \partial^k(\partial_j A^j) = 4\pi q\left[(\rho u^k)_{(+)} - (\rho u^k)_{(-)}\right] \tag{4-4}$$

where $\Box$ is the d'Alembertian operator defined in (1-6). The choice of gauge in (4-4) has been left arbitrary.

Let $\hat{F}^{jk}$ be the dual to F^{jk}. It is defined as follows:

$$\hat{F}^{jk} \equiv \frac{1}{2}\varepsilon^{jk\ell n} F_{\ell n} = \varepsilon^{jk\ell n}\partial_\ell A_n \tag{4-5}$$

where $\varepsilon^{jk\ell n}$ is the antisymmetric unit tensor defined at the end of Section I. Using (2-20) we have

$$\begin{aligned} (\hat{F}^{10}, \hat{F}^{20}, \hat{F}^{30}) &= (B_x, B_y, B_z) = \vec{B} \\ (\hat{F}^{23}, \hat{F}^{31}, \hat{F}^{12}) &= (E_x, E_y, E_z) = \vec{E} \\ \hat{F}^{jk} &= -\hat{F}^{kj} \end{aligned} \tag{4-6}$$

From (4-5) we have

$$\partial_j \hat{F}^{jk} = \varepsilon^{jk\ell n}\partial_j\partial_\ell A_n = 0 \tag{4-7}$$

Using (4-6), we have for the three-dimensional form of (4-7)

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{4-8a}$$

$$\vec{\nabla}\cdot\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \tag{4-8b}$$

which, together with (4-2), constitute Maxwell's equations in three-dimensional form. We note that the equations (4-8) follow directly from the introduction of the 4-vector potential; that is, these equations are the justification for introducing the 4-vector potential.

We remark, incidentally, that using (4-5), it is easy to show that (4-7) can be written in the following alternative form:

$$\partial^j F^{k\ell} + \partial^k F^{\ell j} + \partial^\ell F^{jk} = 0 \tag{4-9}$$

We note that, because of the antisymmetry of F^{jk}, the left side of this equation vanishes identically when any two indices have the same value.

Let E^{jk} be the electromagnetic stress-energy tensor. As is well-known, it has the following form:

$$E^{jk} = \frac{1}{4\pi}\left[F^{j\ell}F_\ell{}^k + \frac{1}{4}g^{jk}(F^{\ell n}F_{\ell n})\right] \tag{4-10}$$

where g^{jk} is the metric tensor specified in (1-1). Thus $4\pi E^{jk}$ may be regarded as the square (in the sense of matrix multiplication) of the field tensor F^{jk}, plus the diagonal tensor $\frac{1}{4} g^{jk}(F^{\ell n} F_{\ell n})$ where, using (2-20), we find

$$\frac{1}{4}(F^{\ell n} F_{\ell n}) = \frac{1}{2}(\vec{B}^2 - \vec{E}^2) \tag{4-11}$$

Using (4-10) and (2-20), we find the following well-known expressions for the elements of E^{jk} :

$$E^{00} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) \equiv \text{electromagnetic energy density} \tag{4-12a}$$

$$\begin{aligned} c(E^{01}, E^{02}, E^{03}) &= c(E^{10}, E^{20}, E^{30}) = \\ &= \frac{c}{4\pi}(\vec{E} \times \vec{B}) \equiv \vec{\mathcal{U}}_{em} \equiv \text{electromagnetic energy flux density (Poynting's vector)} \end{aligned} \tag{4-12b}$$

$$E^{jk} = -\left[E^{00} g^{jk} + \frac{1}{4\pi}(E^j E^k + B^j B^k)\right] \equiv \text{electromagnetic stress tensor} \tag{4-12c}$$

$$(j, k = 1, 2, 3)$$

where

$$(E^1, E^2, E^3) \equiv (E_x, E_y, E_z) \qquad \text{(4-12d)}$$
$$(B^1, B^2, B^3) \equiv (B_x, B_y, B_z)$$

Using the fact that

$$g^j{}_j = 4 \qquad \text{(4-13)}$$

it is evident from (4-10) that

$$E^j{}_j = 0 \qquad \text{(4-14)}$$

Using (4-1), (4-9), and (4-10), we find the following well-known relation:

$$\partial_\ell E^{k\ell} = -q F^{k\ell}\left[(\rho u_\ell)_{(+)} - (\rho u_\ell)_{(-)}\right] \qquad \text{(4-15)}$$

The time-like part of this, which is the statement of the conservation of energy, can be written as follows:

$$\frac{\partial E^{00}}{\partial t} + \vec{\nabla}\cdot\vec{\mathcal{U}}_{em} = -q\vec{E}\cdot\left[(\rho u^0\vec{v})_{(+)} - (\rho u^0\vec{v})_{(-)}\right] \qquad \text{(4-16)}$$

The expression on the right side represents the energy transferred per unit time per unit volume from the charged fluids to the electromagnetic field.

Gravitational Field

We postulate the following source equation for the scalar gravitational potential γ:

$$\Box \gamma = -4\pi \Gamma \left[(\rho m)_{(+)} + (\rho m)_{(-)}\right] \tag{4-17}$$

where $\Gamma = 6.668 \times 10^{-8}$ dyne cm^2/gm^2 is the gravitational constant.

For the time-independent case in which the mass on the right is distributed with spherical symmetry, we find for $(-\vec{\nabla}\gamma)$, which is the gravitational force per unit mass, the following:

$$-\vec{\nabla}\gamma = -\frac{\Gamma m_s}{r^2}\vec{r}_1 \tag{4-18}$$

where $\vec{r}_1$ is the unit radial vector, and m_s is the total mass contained within the sphere of radius r. This, of course, is just the familiar Newtonian gravitational force. Thus (4-17) corresponds to a suitably covariant extension of Newtonian gravitation.

Because we are employing scalar theory, however, rather than a tensor theory as in General Relativity (in which the symmetric metric tensor effectively plays the role of a gravitational potential), we must expect to find that the velocity dependence of the gravitational force on a moving mass is not correct to higher orders of v/c. Moreover, since only the rest-mass appears on the right side of (4-17), our scalar theory neglects the gravitational effect of the mass arising from the thermal, electromagnetic, and gravitational energy of the electron and proton gases. However, these errors are negligible in most magnetogasdynamical problems of physical interest.

Let G^{jk} be the stress-energy tensor of the scalar gravitational field. It has the following form:

$$G^{jk} = \frac{1}{4\pi\Gamma}\left[(\partial^j g)(\partial^k g) - \frac{1}{2} g^{jk}(\partial^\ell g)(\partial_\ell g)\right] \tag{4-19}$$

That this is indeed the correct expression for G^{jk} can be verified by calculating $\partial_j G^{jk}$:

$$\begin{aligned}\partial_\ell G^{k\ell} &= \frac{1}{4\pi\Gamma}\left[(\partial_j \partial^j g)\partial^k g + (\partial_j g)(\partial^j \partial^k g) - (\partial_\ell g)(\partial^k \partial^\ell g)\right] \\ &= \frac{1}{4\pi\Gamma}\left[(\Box g)\partial^k g\right] = -\left[(\rho m)_{(+)} + (\rho m)_{(-)}\right]\partial^k g\end{aligned} \tag{4-20}$$

The right side of (4-20) is just the negative of the gravitational force density acting on the two fluid fields. This is completely analogous with (4-15) in which the right side is the negative of the electromagnetic force density acting on the two charged fluid fields.

From (4-19) we find the following expressions for the elements of G^{jk}:

$$G^{00} = \frac{1}{8\pi\Gamma}\left[\frac{1}{c^2}\left(\frac{\partial g}{\partial t}\right)^2 + (\vec{\nabla} g)^2\right] \tag{4-21a}$$

$\equiv$ gravitational energy density

$$c(G^{01}, G^{02}, G^{03}) = c(G^{10}, G^{20}, G^{30}) = -\frac{1}{4\pi\Gamma}\left(\frac{\partial g}{\partial t}\right)(\vec{\nabla} g) \qquad (4\text{-}21b)$$

$$\equiv \vec{\mathcal{U}}_{gr} \equiv \text{ gravitational energy flux density}$$

$$G^{jk} = \frac{1}{4\pi\Gamma}\left\{\frac{1}{2}\left[(\vec{\nabla} g)^2 + \frac{1}{c^2}\left(\frac{\partial g}{\partial t}\right)^2\right] g^{jk} + (\partial^j g)(\partial^k g)\right\} \qquad (4\text{-}21c)$$

$$(j, k = 1, 2, 3) \qquad \equiv \text{ gravitational stress tensor}$$

From (4-13) and (4-19) we find

$$G^j{}_j = -\frac{1}{4\pi\Gamma}(\partial^\ell g)(\partial_\ell g) = \frac{1}{4\pi\Gamma}\left[(\vec{\nabla} g)^2 - \frac{1}{c^2}\left(\frac{\partial g}{\partial t}\right)^2\right] \qquad (4\text{-}22)$$

Finally, we note that the time-like component of (4-20), which is the statement of conservation of energy, can be written as follows:

$$\frac{\partial G^{00}}{\partial t} + \vec{\nabla}\cdot\vec{\mathcal{U}}_{gr} = -\left[(\rho m)_{(+)} + (\rho m)_{(-)}\right]\frac{\partial g}{\partial t} \qquad (4\text{-}23)$$

where the right-hand side is just the energy transferred per unit time per unit volume <u>from</u> the two fluids <u>to</u> the gravitational field.

V SPIN EFFECTS

As was noted in Section I, we must take particle spin into account, not because it is physically significant in magnetogas-dynamical problems, but rather because it is the price we must pay for the benefits of using the spinor formulation of the problem.

Magnetic Moment

Spin makes itself felt in two ways: through its magnetic moment, and through its angular momentum. We shall consider first the effects of the magnetic moment.

Let the direction of the magnetic moment of a particle in its rest-frame be specified by the 3-vector $\vec{\mu}_{(\pm)}$, which can be generalized to the 4-vector $\mu^{j}_{(\pm)}$ by specifying that the time-like component be zero in the particle rest-frame. We specify the magnitude of the magnetic moment of a particle by the scalar $\mu_{(\pm)}$. (We allow the possibility that $\mu_{(\pm)}$ is not a constant, but rather a function of the fluid and field variables. In Section VIII we shall see that the functional form of $\mu_{(\pm)}$ is automatically specified by the spinor equations.)

The magnetic moment of a particle may also be described in terms of the antisymmetric tensor $(\mu s^{jk})_{(\pm)}$ where $s^{jk}_{(\pm)}$ is derived from $\mu^j_{(\pm)}$ and the particle 4-velocity $u^j_{(\pm)}$ as follows:

$$s^{jk}_{(\pm)} \equiv \varepsilon^{jkln}(\mu_l u_n)_{(\pm)} \tag{5-1}$$

Thus

$$[\rho\mu(s^{23}, s^{31}, s^{12})]_{(\pm)} = [(\rho u^0)(\mu\vec{\mu})]_{(\pm)} \tag{5-2a}$$

$$[\rho\mu(s^{10}, s^{20}, s^{30})]_{(\pm)} = \{(\rho u^0)[\tfrac{1}{c}\vec{v}\times(\mu\mathring{\vec{\mu}})]\}_{(\pm)} \tag{5-2b}$$

where in (5-2b) we have made use of the fact that $\vec{v}\times\vec{\mu} = \vec{v}\times\mathring{\vec{\mu}}$, which follows directly from the form of the Lorentz transformation.

Since ρu^0 is just the particle density in the observer's frame, (5-2a) tells us that the space-space part of $\rho\mu s^{jk}$ is just the magnetization of the fluid. We note that the cross-product inside the brackets on the right side of (5-2b) is just the well-known electric dipole associated with a moving magnetic dipole. Thus, even though electrons and protons have no electric dipole moment at rest, a spin-dependent electric polarization of the fluid is possible when there is motion.

Now we know that in a magnetized body there is an effective current density given by the curl of the magnetization. By analogy, we would expect a 4-momentum density to be associated with the angular momentum density arising from particle spin. If $\delta p^j_{(\pm)}$

is this extra 4-momentum density divided by ρ, i.e. the extra momentum per particle, then (3-4) must be replaced by

$$-\partial^j \Phi_{(\pm)} = (\breve{m} c u^j)_{(\pm)} \pm \frac{q}{c} A^j + \delta p^j_{(\pm)} \tag{5-3}$$

The form of this equation indicates that it should be possible to absorb $\delta p^j_{(\pm)}$ into the 4-potential, i.e., to express the spin effects as a small correction to the electromagnetic field. In order to be thorough about this, we must keep in mind that every particle has a contribution to its rest-mass $\Delta m_{(\pm)}$ arising from the interaction of the magnetic field and the magnetic dipole moment. If $\overset{\circ}{\vec{B}}$ and $(\mu \overset{\circ}{\vec{s}})_{(\pm)}$ are respectively the magnetic field and the magnetic dipole moment in the particle rest-frame, then the magnetic contribution to the particle rest-mass is

$$\Delta m_{(\pm)} = -\frac{1}{c^2} (\mu \overset{\circ}{\vec{s}})_{(\pm)} \cdot \overset{\circ}{\vec{B}} = \frac{1}{2c^2} (\mu s^{jk})_{(\pm)} F_{jk} \tag{5-4}$$

where, in the second step, we have made use of (2-20) and (5-2).

If we wish to express the spins effects in terms of a small correction $a^j_{(\pm)}$ to the 4-vector potential A^j, then we must introduce a corresponding correction $f^{jk}_{(\pm)}$ to the field tensor F^{jk} where, in analogy to (4-3), we have

$$f^{jk}_{(\pm)} = \partial^j a^k_{(\pm)} - \partial^k a^j_{(\pm)} \tag{5-5}$$

Analogous to (2-20), we can define corrections $\vec{e}_{(\pm)}$ and $\vec{b}_{(\pm)}$ to the electric and magnetic field intensities $\vec{E}$ and $\vec{B}$ as follows:

$$\begin{aligned} (f^{10}, f^{20}, f^{30})_{(\pm)} &\equiv (e_x, e_y, e_z)_{(\pm)} = \vec{e}_{(\pm)} \\ (f^{23}, f^{31}, f^{12})_{(\pm)} &\equiv -(b_x, b_y, b_z)_{(\pm)} = -\vec{b}_{(\pm)} \\ f^{jk}_{(\pm)} &= -f^{kj}_{(\pm)} \end{aligned} \tag{5-6}$$

If $f^{jk}_{(\pm)}$ is to be regarded as completely analogous to F^{jk}, then it should produce a contribution $\delta m_{(\pm)}$ to the particle rest-mass having the form of (5-4) with F_{jk} replaced by $f^{(\pm)}_{jk}$:

$$\delta m_{(\pm)} = -\frac{1}{c^2}\left[(\mu\mathring{\vec{\mu}})\cdot\mathring{\vec{b}}\right]_{(\pm)} = \frac{1}{2c^2}(\mu s^{jk} f_{jk})_{(\pm)} \tag{5-7}$$

Let $\widetilde{m}_{(\pm)}$ be the total magnetic contribution to the particle rest-mass. Then

$$\begin{aligned} \widetilde{m}_{(\pm)} = \Delta m_{(\pm)} + \delta m_{(\pm)} &= -\frac{1}{c^2}\left[(\mu\mathring{\vec{\mu}})\cdot(\mathring{\vec{B}}+\mathring{\vec{b}})\right]_{(\pm)} \\ &= \frac{1}{2c^2}(\mu s_{jk})_{(\pm)}(F^{jk}+f^{jk}_{(\pm)}) \end{aligned} \tag{5-8}$$

If we regard $u^j_{(\pm)}$, $(\mu s^{jk})_{(\pm)}$, and $\delta p^j_{(\pm)}$ as known functions, then it is evident from (5-3) that $a^j_{(\pm)}$ is determined by the condition

$$\delta p^j_{(\pm)} = (\widetilde{m} c u^j)_{(\pm)} \pm \frac{q}{c} a^j_{(\pm)} \tag{5-9}$$

where from (5-5) and (5-8) we see that $\tilde{m}_{(\pm)}$ is a function of the derivatives of $a^{j}_{(\pm)}$. Thus (5-9) is a first-order differential equation for $a^{j}_{(\pm)}$. In actual fact, it will not be necessary to solve such an equation since, as we shall see in Section VIII, the spinor equations give us an expression for $a^{j}_{(\pm)}$ that automatically satisfies (5-9).

If we let $M_{(\pm)}$ be the total particle rest-mass, we have from (2-17) and (5-8)

$$\begin{aligned} M_{(\pm)} &= \breve{m}_{(\pm)} + \tilde{m}_{(\pm)} \\ &= m_{(\pm)}\left(1 + g/c^2 + h_{(\pm)}/c^2\right) + \frac{1}{2c^2}\left(\mu\nu_{jk}\right)_{(\pm)}\left[F^{jk} + f^{jk}_{(\pm)}\right] \end{aligned} \tag{5-10}$$

Substituting (5-9) and (5-10) into (5-3), we obtain

$$-\partial^{j}\Phi = \left(Mcu^{j}\right)_{(\pm)} \pm \frac{q}{c}\left[A^{j} + a^{j}_{(\pm)}\right] \tag{5-11}$$

This way of formulating the dynamical effects of spin gives us an immediate intuitive feeling for these effects. Moreover it provides a neat way of determining when these effects can be neglected: If we can show that $|a^{j}_{(\pm)}| \ll |A^{j}|$, then we know that the dynamical effects of spin are negligible. We shall see in Section VIII that it is very easy to estimate the magnitude of the $a^{j}_{(\pm)}$ that results from the spinor equations, and that, in all magnetogas-dynamical problems on a laboratory or astronomical scale, $a^{j}_{(\pm)}$ is completely negligible compared with the magnitudes of A^{j}

encountered in such problems. If, however, we were to attempt to apply our spinor formulation of magnetogas dynamics on an atomic or nuclear scale, then it turns out that the spin effects represented by the effective potential $a^j_{(\pm)}$ would be far from negligible.

We note that in general $a^j_{(+)} \neq a^j_{(-)}$. This is not surprising since the correction potentials $a^j_{(+)}$ and $a^j_{(-)}$ have their origins in dynamical effects of the spin fields $(\rho\mu^j)_{(+)}$ and $(\rho\mu^j)_{(-)}$ which are in general different. From a purely formal point of view, the equations that follow treat $a^j_{(+)}$, for example, as an electromagnetic field which produces a self-interaction within the proton gas, but which produces no interaction between the electron and proton gases. Similarly for $a^j_{(-)}$. In this respect, the effects of $a^j_{(\pm)}$ are analogous to the behavior of the exchange forces in quantum mechanics which act only between identical particles, and not between unlike particles.

To complete the derivation of the dynamical equations including the effects of spin, we take the curl of (5-11), and arrive at the following:

$$\partial^j(Mcu^k)_{(\pm)} - \partial^k(Mcu^j)_{(\pm)} = \mp \frac{q}{c}\left[F^{jk} + f^{jk}_{(\pm)}\right] \qquad (5\text{-}12)$$

This is just the generalization of the tensor equation (2-40). Contracting (5-12) with $u^{(\pm)}_j$, we arrive at the following generalization of the Euler equation in the form (2-24):

$$\left[\frac{d(Mcu^k)}{d\tau}\right]_{(\pm)} = \partial^k(Mc^2)_{(\pm)} \pm q\left[F^{k\ell} + f^{k\ell}_{(\pm)}\right] u_\ell^{(\pm)} \tag{5-13}$$

<u>Fluid Stress-Energy Tensor</u>

As a preliminary to introducing the stress-energy tensor for each of the two fluid fields, we multiply (5-13) by $\rho_{(\pm)}$ and write it in the following form:

$$\left[\rho\frac{d(\breve{m}cu^j)}{d\tau}\right]_{(\pm)} + \left[\rho\frac{d(\tilde{m}cu^j)}{d\tau} - \rho\partial^j(\tilde{m}c^2) \mp q\rho f^{j\ell}u_\ell\right]_{(\pm)} = k^j_{(\pm)} \tag{5-14}$$

where $k^j_{(\pm)}$ is the force density acting on the proton or electron gas:

$$k^j_{(\pm)} = (\rho m)_{(\pm)}\partial^j g + \partial^j P_{(\pm)} \pm qF^{j\ell}(\rho u_\ell)_{(\pm)} = \rho_{(\pm)}\partial^j(\breve{m}c^2)_{(\pm)} \pm qF^{j\ell}(\rho u_\ell)_{(\pm)} \tag{5-15}$$

We define the fluid stress-energy tensor $T^{jk}_{(\pm)}$ for either the proton or electron gas as follows:

$$T^{jk}_{(\pm)} = (\rho\breve{m}c^2u^ju^k)_{(\pm)} + t^{jk}_{(\pm)} \tag{5-16}$$

where the tensor $t^{jk}_{(\pm)}$ contains all the spin dependences. We impose the following condition on $t^{jk}_{(\pm)}$:

$$\partial_k t^{jk}_{(\pm)} = \left[\rho\frac{d(\tilde{m}cu^j)}{d\tau} - \rho\partial^j(\tilde{m}c^2) \mp q\rho f^{j\ell}u_\ell\right]_{(\pm)} \tag{5-17}$$

(In Section VIII we shall see that the spinors equations specify the functional dependence of $t^{jk}_{(\pm)}$ in such a way that condition (5-17) is automatically satisfied.)

We augment (5-17) with the continuity condition:

$$\partial_j (\rho u^j)_{(\pm)} = 0 \tag{5-18}$$

These two conditions, together with the definitions (5-16), suffice to guarantee the validity of the following relation:

$$\partial_\ell T^{j\ell}_{(\pm)} = k^j_{(\pm)} \tag{5-19}$$

This equation, which is identical with (5-13) when (5-17) and (5-18) are taken into account, is the justification for regarding the tensor defined in (5-16) as the fluid stress-energy tensor.

We now make the following identifications:

$$\text{fluid energy density} \equiv T^{00}_{(\pm)} \tag{5-20a}$$

$$\text{fluid energy flux density} \equiv \vec{\mathcal{U}}_{(\pm)} \equiv c\,(T^{01}, T^{02}, T^{03})_{(\pm)} \tag{5-20b}$$

$$\text{fluid momentum density} \equiv \vec{\mathcal{G}}_{(\pm)} \equiv (\mathcal{G}^1, \mathcal{G}^2, \mathcal{G}^3)_{(\pm)} \equiv \frac{1}{c}(T^{10}, T^{20}, T^{30})_{(\pm)} \tag{5-20c}$$

In terms of these quantities, (5-19) can be written as follows:

$$\frac{\partial T^{00}_{(\pm)}}{\partial t} + \vec{\nabla}\cdot\vec{\mathcal{U}}_{(\pm)} = c\,k^0_{(\pm)} \tag{5-21a}$$

$$\frac{\partial \mathcal{G}^{j}_{(\pm)}}{\partial t} = -\partial_\ell T^{j\ell}_{(\pm)} + k^{j}_{(\pm)} \qquad (j, \ell = 1, 2, 3) \tag{5-21b}$$

These equations are the justification for the identifications made in (5-20).

Fluid Angular Momentum Tensor

If the spin angular momentum density of either the proton or electron gas is designated by $\mathcal{L}^{jk\ell}_{(\pm)}$, and the total angular momentum density of either charged gas by $M^{jk\ell}_{(\pm)}$, then

$$M^{jk\ell}_{(\pm)} \equiv \mathcal{L}^{jk\ell}_{(\pm)} + \frac{1}{c}\left[x^{j}\, T^{k\ell}_{(\pm)} - x^{k}\, T^{j\ell}_{(\pm)}\right] \tag{5-22}$$

The intuitive significance of this definition is more apparent if it is written in the following form:

$$\vec{M}_{(\pm)} = \vec{\mathcal{L}}_{(\pm)} + \vec{r} \times \vec{\mathcal{G}}_{(\pm)} \tag{5-23}$$

where

$$\vec{M}_{(\pm)} = (M_x, M_y, M_z)_{(\pm)} \equiv (M^{230}, M^{310}, M^{120})_{(\pm)} \tag{5-24}$$

$$\vec{\mathcal{L}}_{(\pm)} = (\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z)_{(\pm)} \equiv (\mathcal{L}^{230}, \mathcal{L}^{310}, \mathcal{L}^{120})_{(\pm)} \tag{5-25}$$

$$\vec{r} = (x, y, z) \equiv (x^1, x^2, x^3) \tag{5-26}$$

and $\vec{\mathcal{G}}_{(\pm)}$ is defined in (5-20c). The 3-vectors $\vec{\mathcal{L}}_{(\pm)}$ and $\vec{M}_{(\pm)}$ are the quantities to be regarded respectively as the spin and total angular momentum densities, but the complete tensors $\mathcal{L}^{jk\ell}_{(\pm)}$ and $M^{jk\ell}_{(\pm)}$ are necessary for a covariant formulation of the theory.

The conservation of angular momentum requires that the following condition be satisfied:

$$\partial_\ell M^{jk\ell}_{(\pm)} = \frac{1}{c}\left[x^j k^k_{(\pm)} - x^k k^j_{(\pm)}\right] \tag{5-27}$$

The integral form of this equation makes its significance more apparent. To derive this we first write the space-space part of (5-27) as follows:

$$\frac{\partial \vec{M}_{(\pm)}}{\partial t} + c\,\partial_\ell \vec{M}^\ell_{(\pm)} = \vec{r} \times \vec{k}_{(\pm)} \qquad (\ell = 1, 2, 3) \tag{5-28a}$$

where

$$\vec{M}^\ell_{(\pm)} = (M^\ell_x, M^\ell_y, M^\ell_z)_{(\pm)} \equiv (M^{23\ell}, M^{31\ell}, M^{12\ell})_{(\pm)} \quad (\ell = 1, 2, 3) \tag{5-28b}$$

$$\vec{k}_{(\pm)} = (k_x, k_y, k_z)_{(\pm)} \equiv (k^1, k^2, k^3)_{(\pm)} \tag{5-28c}$$

We integrate (5-28a) over the 3-volume V_3 contained within a large sphere of radius R. Since the term involving $\vec{M}^\ell_{(\pm)}$ is a 3-divergence, it can be transformed by means of Gauss' theorem into an integral over the surface of the sphere. If the two charged fluids are contained completely within the sphere, or if their distribution is such that as $R \to \infty$, the magnitude of $\vec{M}^\ell$ on the surface of the sphere falls off faster than R^{-2} then the integral over all 3-space of the terms in (5-28a) involving $\vec{M}^{\ell}$ will vanish and we are led to the desired result:

$$\frac{d}{dt}\int_{V_3} \vec{M}_{(\pm)}\, dV_3 = \int_{V_3} (\vec{r} \times \vec{k}_{(\pm)})\, dV_3 \tag{5-29}$$

where the integration extends over all 3-space. This equation states that the time rate of change of the total angular momentum of one of the charged gases equals the total torque acting on the gas.

Substituting (5-22) into (5-27) and using (5-19), we arrive at the following condition on $\mathcal{L}^{jk\ell}_{(\pm)}$:

$$\partial_\ell \mathcal{L}^{jk\ell}_{(\pm)} = \frac{1}{c}(T^{jk} - T^{kj})_{(\pm)} = \frac{1}{c}(t^{jk} - t^{kj})_{(\pm)} \tag{5-30}$$

This shows that when a fluid has spin, its stress-energy tensor cannot, in general, be completely symmetric. We shall see in Section VIII that the spinor equations specify the functional forms of $\mathcal{L}^{jk\ell}_{(\pm)}$ and $t^{jk}_{(\pm)}$ in such a way that the condition (5-30) is automatically satisfied.

Total Stress-Energy Tensor

From (5-15) and (5-19) we have

$$\begin{aligned}\partial_\ell[T^{j\ell}_{(+)} + T^{j\ell}_{(-)}] &= \partial^j[P_{(+)} + P_{(-)}] + [(\rho m)_{(+)} + (\rho m)_{(-)}]\,\partial^j g \\ &\quad + q\, F^{j\ell}[(\rho u_\ell)_{(+)} - (\rho u_\ell)_{(-)}] \\ &= \partial^j[P_{(+)} + P_{(-)}] - \partial_\ell G^{j\ell} - \partial_\ell E^{j\ell}\end{aligned} \tag{5-31}$$

where we have made use of (4-15) and (4-20). Thus

$$\partial_\ell \mathcal{T}^{j\ell} = 0 \tag{5-32}$$

where

$$\begin{aligned} \mathcal{T}^{j\ell} &= T^{j\ell}_{(+)} + T^{j\ell}_{(-)} - [P_{(+)} + P_{(-)}]\, g^{j\ell} + G^{j\ell} + E^{j\ell} \\ &= (\rho \breve{m} c^2 u^j u^\ell - P g^{j\ell})_{(+)} + (\rho \breve{m} c^2 u^j u^\ell - P g^{j\ell})_{(-)} + t^{j\ell}_{(+)} + t^{j\ell}_{(-)} + G^{j\ell} + E^{j\ell} \end{aligned} \tag{5-33}$$

Obviously, $\mathcal{T}^{j\ell}$ is the total stress-energy tensor of the entire magnetogas-dynamical system including charged fluids and fields.

We note, incidentally, that using (2-1), (2-13), and (2-17) we have

$$(\rho \breve{m} c^2 u^j u^\ell - P g^{j\ell})_{(\pm)} = [(\varepsilon + P) u^j u^\ell - P g^{j\ell}]_{(\pm)} \tag{5-34}$$

where

$$\varepsilon_{(\pm)} = [\rho (mc^2 + m\mathcal{g} + m\mathcal{U})]_{(\pm)} \tag{5-35}$$

is the energy density of the fluid in its rest-frame. The right side of (5-34) is the usual relativistic form for the stress-energy tensor of a perfect fluid (viscosity neglected).*

*See, for example, Möller (ref. 10) p. 182, eq. 104

We now make the following identifications:

$$\text{total energy density} \equiv T^{00} \tag{5-36a}$$

$$\begin{aligned}\text{total energy flux density} \equiv \vec{\mathcal{U}} &\equiv c(T^{01}, T^{02}, T^{03}) \\ &= \vec{\mathcal{U}}_{(+)} + \vec{\mathcal{U}}_{(-)} + \vec{\mathcal{U}}_{em} + \vec{\mathcal{U}}_{gr}\end{aligned} \tag{5-36b}$$

$$\begin{aligned}\text{total momentum density} \equiv \vec{\mathcal{G}} &= (\mathcal{G}^1, \mathcal{G}^2, \mathcal{G}^3) \\ &\equiv \frac{1}{c}(T^{10}, T^{20}, T^{30}) \\ &= \vec{\mathcal{G}}_{(+)} + \vec{\mathcal{G}}_{(-)} + \vec{\mathcal{G}}_{em} + \vec{\mathcal{G}}_{gr}\end{aligned} \tag{5-36c}$$

In terms of these quantities (5-32) can be written as follows:

$$\frac{\partial T^{00}}{\partial t} + \vec{\nabla} \cdot \vec{\mathcal{U}} = 0 \tag{5-37a}$$

$$\frac{\partial \mathcal{G}^j}{\partial t} = -\partial_\ell T^{j\ell} \quad (j, \ell = 1, 2, 3) \tag{5-37b}$$

These equations, of course, are the justification for the identifications made in (5-36)

We can integrate the equations (5-37) in just the same way we integrated (5-28a). Doing this, we find

$$\mathcal{E} = \int_{V_3} T^{00} dV_3 = \text{constant} \tag{5-38}$$

$$\vec{\mathcal{P}} = \int_{V_3} \vec{\mathcal{G}}\, dV_3 = \text{constant} \tag{5-39}$$

where $\mathcal{E}$ is the total energy of the entire magnetogas-dynamical system, and $\vec{\mathcal{P}}$ is the total momentum.

Total Angular Momentum Tensor

If $M^{jk\ell}$ is the total angular momentum density tensor for the entire magnetogas-dynamical system, we have

$$M^{jk\ell} \equiv \mathcal{L}^{jk\ell}_{(+)} + \mathcal{L}^{jk\ell}_{(-)} + \frac{1}{c}\left(x^j T^{k\ell} - x^k T^{j\ell}\right) \tag{5-40}$$

The condition for conservation of angular momentum is

$$\begin{aligned} 0 = \partial_\ell M^{jk\ell} &= \partial_\ell\left[\mathcal{L}^{jk\ell}_{(+)} + \mathcal{L}^{jk\ell}_{(-)}\right] + \frac{1}{c}\left(T^{kj} - T^{jk}\right) \\ &= \left[\partial_\ell \mathcal{L}^{jk\ell} - \frac{1}{c}\left(t^{jk} - t^{kj}\right)\right]_{(+)} + \left[\partial_\ell \mathcal{L}^{jk\ell} - \frac{1}{c}\left(t^{jk} - t^{kj}\right)\right]_{(-)} \end{aligned} \tag{5-41}$$

where we have used the fact that $t^{jk}_{(+)}$ and $t^{jk}_{(-)}$ are the only parts of T^{jk} that contain any antisymmetry. Obviously, if the condition (5-30) is satisfied, conservation of total angular momentum follows automatically from the definition (5-40).

Integrating (5-41) over all 3-space in the same way we integrated (5-27), we have the following result:

$$\mathcal{M}^{jk} = \int_{V_3} M^{jk0}\, dV_3 = \text{constant} \tag{5-42}$$

where $\mathcal{M}^{jk}$ is the total angular momentum of the entire system. If we define the 3-vector momentum $\vec{\mathcal{M}}$ as

$$\vec{\mathcal{M}} = (\mathcal{M}_x, \mathcal{M}_y, \mathcal{M}_z) \equiv (\mathcal{M}^{23}, \mathcal{M}^{31}, \mathcal{M}^{12}) \tag{5-43}$$

then from (5-42), (5-41), (5-36c) and (5-25) we have

$$\vec{\mathcal{M}} = \int_{V_3} \vec{\mathrm{M}}\, dV_3 = \int_{V_3} [\vec{\mathcal{L}}_{(+)} + \vec{\mathcal{A}}_{(-)} + \vec{r} \times \vec{\mathcal{S}}]\, dV_3 \quad = \quad \text{constant} \tag{5-44}$$

It is possible to express the total angular momentum in terms of a 4-vector Σ^j instead of the antisymmetric tensor $\mathcal{M}^{jk}$ once we have defined the 4-velocity $\mathcal{V}^j$ of the entire system.

If $\vec{V}$ is the 3-velocity corresponding to $\mathcal{V}^j$, then by definition

$$\mathcal{V}^0 \equiv [1 - \vec{V}^2/c^2]^{-1/2} \tag{5-45a}$$

$$\mathcal{V}^j \equiv \frac{\mathcal{V}^0}{c} V^j \qquad (j = 1, 2, 3) \tag{5-45b}$$

$$\mathcal{V}^j \mathcal{V}_j = 1 \tag{5-45c}$$

$\mathcal{V}^j$ is defined in terms of $\mathcal{E}$ and $\vec{p} = (p^1, p^2, p^3)$ as follows:

$$\mathcal{V}^0 \equiv \mathcal{E}[\mathcal{E}^2 - c^2\vec{p}^2]^{-1/2} \tag{5-46a}$$

$$\mathcal{V}^j \equiv c\, p^j [\mathcal{E}^2 - c^2\vec{p}^2]^{-1/2} \qquad (j = 1, 2, 3) \tag{5-46b}$$

From (5-45) and (5-46), it follows that

$$\vec{V} = \frac{\vec{p}}{(\mathcal{E}/c^2)} \tag{5-47}$$

where the quantity in the denominator on the right is just the total mass of the system in the observer's reference frame. The square root in the denominator of (5-46) corresponds to the rest-energy of the entire system.

Using the 4-velocity defined in (5-46), we define the 4-vector angular momentum Σ^j as follows:

$$\Sigma^j \equiv -\frac{1}{2}\,\varepsilon^{jk\ell n}\,v_k\,\mathcal{M}_{\ell n} \tag{5-48}$$

In the rest-frame of the total system $(\vec{V}=0)$, we have

$$\left.\begin{aligned} \mathring{\Sigma}^0 &= 0 \\ (\mathring{\Sigma}^1, \mathring{\Sigma}^2, \mathring{\Sigma}^3) &= \vec{\mathring{\mathcal{M}}} \end{aligned}\right\} \text{for } \vec{V}=0 \tag{5-49}$$

VI SUMMARY OF FOREGOING RESULTS

We have seen that the dynamical side of the magnetogas-dynamical problem can be specified by the following system of equations:

$$\left[\frac{d(Mcu^j)}{d\tau}\right]_{(\pm)} = \partial^j(Mc^2)_{(\pm)} \pm q\left[F^{jk} + f^{jk}_{(\pm)}\right]u_k^{(\pm)} \tag{6-1a}$$

$$\partial_j(\rho u^j)_{(\pm)} = 0 \tag{6-1b}$$

$$(u_j u^j)_{(\pm)} = 1 \tag{6-1c}$$

where

$$M_{(\pm)} = \breve{m}_{(\pm)} + \tilde{m}_{(\pm)} \tag{6-1d}$$

$$\breve{m}_{(\pm)} = m_{(\pm)}\left[1 + g/c^2 + h_{(\pm)}/c^2\right] \tag{6-1e}$$

$$\tilde{m}_{(\pm)} = \frac{1}{2c^2}(\mu s_{jk})_{(\pm)}\left[F^{jk} + f^{jk}_{(\pm)}\right] \tag{6-1f}$$

We impose the adiabatic condition

$$\left[\rho\,\partial_j h - (\gamma-1)h\,\partial_j\rho\right]_{(\pm)} = 0 \tag{6-2a}$$

or

$$h_{(\pm)} = \left[H(\rho/\rho_0)^{(\gamma-1)}\right]_{(\pm)} \tag{6-2b}$$

where

$$\rho^{\circ}_{(\pm)} = \left[\rho(x^j_{(0)})\right]_{(\pm)} \quad ; \qquad H_{(\pm)} = \left[h(x^j_{(0)})\right]_{(\pm)} \qquad (6\text{-}2c)$$

where $x^j_{(0)}$ are the coordinates of a particular point in space-time. If the condition (6-2) is satisfied, then

$$\left[\rho\, \partial^j (Mc^2)\right]_{(\pm)} = (\rho m)_{(\pm)} \partial^j \varphi + \partial^j P_{(\pm)} + \left[\rho\, \partial^j (\tilde{m} c^2)\right]_{(\pm)} \quad (6\text{-}3)$$

It is this relation which allows us to replace the force densities in Euler's equation that are associated with gravitation, pressure, and spin-field interaction by the particle density times the gradient of the total particle rest-energy $M_{(\pm)}c^2$.

The specific enthalpy $h_{(\pm)}$ is the only thermodynamic function we need know, since the other two thermodynamic functions of principal interest, $T_{(\pm)}$ and $P_{(\pm)}$, can be found from the following relations:

$$T_{(\pm)} = \left(h/c_p\right)_{(\pm)} \qquad (6\text{-}4)$$

and

$$P_{(\pm)} = \left[(\rho m h)\left(1 - \frac{1}{\gamma}\right)\right]_{(\pm)} \qquad (6\text{-}5)$$

We have seen that there exists a simplified alternative to the system (6-1):

$$-\partial^j \Phi_{(\pm)} = (Mc\, u^j)_{(\pm)} \pm \frac{q}{c}\left[A^j + a^j_{(\pm)}\right] \qquad (6\text{-}6a)$$

$$\partial_j(\rho u^j)_{(\pm)} = 0 \tag{6-6b}$$

$$(u_j u^j)_{(\pm)} = 0 \tag{6-6c}$$

Taking the curl of (6-9a) yields the tensor alternative to Euler's equation:

$$\partial^j(Mcu^k)_{(\pm)} - \partial^k(Mcu^j)_{(\pm)} = \mp \frac{q}{c}\left[F^{jk} + f^{jk}_{(\pm)}\right] \tag{6-7}$$

Contracting (6-7) with $u_j^{(\pm)}$ yields Euler's equation in the form (6-1a). Thus any solution of (6-7) or (6-6a) is also a solution of (6-1a), but the converse is not in general true. Thus (6-6a) and (6-7) correspond to a more restricted class of solutions than (6-1a). The source of this restriction can be stated in either of two ways: (1) The restriction is just the requirement that there can be no local vorticity in either charged gas without a magnetic field to provide the necessary Coriolis force arising from the random thermal motion; or (2) The restriction results from the requirement (3-2) that the generalized particle momentum must be the gradient of a scalar, which is just the incorporation of the de Broglie Hypothesis into the formalism.

Obviously a choice must be made between the systems (6-1) and (6-6), and we have chosen in favor of (6-6) which is the more fundamental formulation of magnetogas dynamics, because the added restriction it imposes is physically necessary for a charged fluid with random thermal motion.

Thus the fluid dynamical half of the problem is specified by the system (6-6). The field half of the problem is specified by the following group of equations:

$$\left. \begin{array}{ll} \left[\rho \partial_j h - (\gamma-1) h \partial_j \rho\right]_{(\pm)} = 0 & \text{(6-8a)} \\ \text{or} & \\ h_{(\pm)} = \left[H(\rho/\rho_0)^{(\gamma-1)}\right]_{(\pm)} & \text{(6-8b)} \end{array} \right\}$$

$$\Box A^j - \partial^j(\partial_k A^k) = 4\pi q\left[(\rho u^j)_{(+)} - (\rho u^j)_{(-)}\right] \tag{6-8c}$$

$$\Box g = -4\pi\Gamma\left[(\rho m)_{(+)} + (\rho m)_{(-)}\right] \tag{6-8d}$$

Equations (6-8a) and (6-8b) are just the equations (6-2), and have been repeated here to emphasize that we are treating $h_{(\pm)}$ as a "thermal field" on a par with the gravitational and electromagnetic fields represented by g and A^j. We have a choice between (6-8a) and (6-8b), and can use the one that is simpler for calculational purposes. We noted in (2-33) - (2-36) that if the constants $H_{(+)}$ and $H_{(-)}$ satisfy the condition

$$(mH)_{(+)} \approx (mH)_{(-)} \tag{6-9}$$

then the electron and proton temperatures are everywhere approximately equal.

The stress-energy tensor $\mathcal{T}^{jk}$ of the entire system, including both charged fluids and the fields, has the following form:

$$\mathcal{T}^{jk} = T^{jk}_{(+)} + T^{jk}_{(-)} - [P_{(+)} + P_{(-)}]\, g^{jk} + E^{jk} + G^{jk} \tag{6-10}$$

where

$$T^{jk}_{(\pm)} = (\rho \breve{m} c^2 u^j u^k)_{(\pm)} + t^{jk}_{(\pm)} \tag{6-11}$$

$$E^{jk} = \frac{1}{4\pi}\left[F^{j\ell} F_\ell{}^k + \frac{1}{4} g^{jk}(F^{\ell n} F_{\ell n})\right] \tag{6-12}$$

$$G^{jk} = \frac{1}{4\pi\Gamma}\left[(\partial^j g)(\partial^k g) - \frac{1}{2} g^{jk}(\partial^\ell g)(\partial_\ell g)\right] \tag{6-13}$$

These tensors satisfy the following divergence relations:

$$\partial_k \mathcal{T}^{jk} = 0 \tag{6-14}$$

$$\partial_k T^{jk}_{(\pm)} = k^j_{(\pm)} \tag{6-15}$$

where

$$\begin{aligned} k^j_{(\pm)} &= (\rho m)_{(\pm)} \partial^j g + \partial^j P_{(\pm)} \pm q F^{jk} (\rho u_k)_{(\pm)} \\ &= [\rho \partial^j (\breve{m} c^2)]_{(\pm)} \pm q F^{jk} (\rho u_k)_{(\pm)} \end{aligned} \tag{6-16}$$

$$\partial_k E^{jk} = -q F^{j\ell}[(\rho u_\ell)_{(+)} - (\rho u_\ell)_{(-)}] \tag{6-17}$$

$$\partial_k G^{jk} = -\left[(\rho m)_{(+)} + (\rho m)_{(-)}\right] \partial^j g \tag{6-18}$$

Equation (6-14) justifies the following identifications:

$$\text{total energy density} \equiv T^{00} \tag{6-19}$$

$$\text{total energy flux density} \equiv \vec{\mathcal{U}} \equiv c(T^{01}, T^{02}, T^{03}) \tag{6-20}$$

$$\text{total momentum density} \equiv \vec{\mathcal{G}} \equiv \frac{1}{c}(T^{10}, T^{20}, T^{30}) \tag{6-21}$$

The invariant trace of the total stress-energy tensor is

$$\begin{aligned} T^j{}_j &= (\rho \breve{m} c^2 + t^j{}_j - 4P)_{(+)} + (\rho \breve{m} c^2 + t^j{}_j - 4P)_{(-)} + \frac{1}{4\pi\Gamma}\left[(\vec{\nabla} g)^2 - \frac{1}{c^2}\left(\frac{\partial g}{\partial t}\right)^2\right] \\ &= \left[(\varepsilon - 3P) + t^j{}_j\right]_{(+)} + \left[(\varepsilon - 3P) + t^j{}_j\right]_{(-)} + \frac{1}{4\pi\Gamma}\left[(\vec{\nabla} g)^2 - \frac{1}{c^2}\left(\frac{\partial g}{\partial t}\right)^2\right] \end{aligned} \tag{6-22}$$

where $\varepsilon_{(\pm)}$ is the fluid energy density defined in (5-35).

If $\mathcal{E}$ and $\vec{\mathcal{P}}$ are respectively the total energy and momentum of a finite system, it follows from (6-14) that

$$\mathcal{E} = \int_{V_3} T^{00} dV_3 = \text{constant} \tag{6-23}$$

$$\vec{\mathcal{P}} = \int_{V_3} \vec{\mathcal{G}} \, dV_3 = \text{constant} \tag{6-24}$$

where the integration extends over all 3-space. The 4-velocity $\mathcal{V}^j$ of the entire system is defined as

$$\mathcal{V}^0 \equiv \mathcal{E}\left[\mathcal{E}^2 - c^2 \vec{\mathcal{P}}^2\right]^{-1/2} \tag{6-25a}$$

$$\mathcal{V}^j \equiv c\,\mathcal{P}^j\left[\mathcal{E}^2 - c^2 \vec{\mathcal{P}}^2\right]^{-1/2} \quad (j = 1, 2, 3) \tag{6-25b}$$

which corresponds to a 3-velocity $\vec{V}$ given by

$$\vec{V} = \frac{\vec{\mathcal{P}}}{(\mathcal{E}/c^2)} \tag{6-26}$$

The square root in the denominator of (6-25) is the total rest-energy of the system, and the factor $(\mathcal{E}/c^2)$ in the denominator of (6-26) is the total mass of the system in the observer's reference frame.

If $\mathcal{S}^{jk\ell}_{(\pm)}$ is spin angular momentum density of either charged fluid, then the total fluid angular momentum $M^{jk\ell}_{(\pm)}$ is defined as follows:

$$M^{jk\ell}_{(\pm)} \equiv \mathcal{S}^{jk\ell}_{(\pm)} + \frac{1}{c}\left[x^j T^{k\ell}_{(\pm)} - x^k T^{j\ell}_{(\pm)}\right] \tag{6-27}$$

This definition can be written in terms of 3-vectors as follows:

$$\vec{M}_{(\pm)} = \vec{\mathcal{S}}_{(\pm)} + \vec{r} \times \vec{\mathcal{G}}_{(\pm)} \tag{6-28}$$

where

$$\vec{M}_{(\pm)} \equiv (M_x, M_y, M_z)_{(\pm)} \equiv (M^{230}, M^{310}, M^{120})_{(\pm)} \tag{6-29}$$

$$\vec{\mathcal{S}}_{(\pm)} = (\mathcal{S}_x, \mathcal{S}_y, \mathcal{S}_z)_{(\pm)} \equiv (\mathcal{S}^{230}, \mathcal{S}^{310}, \mathcal{S}^{120})_{(\pm)} \tag{6-30}$$

$$\vec{r} = (x, y, z) \equiv (x^1, x^2, x^3) \tag{6-31}$$

Conservation of fluid angular momentum requires that

$$\partial_\ell M^{jk\ell}_{(\pm)} = \frac{1}{c}\left[x^j k^k_{(\pm)} - x^k k^j_{(\pm)}\right] \tag{6-32}$$

where $\ell^{j}_{(\pm)}$ is given by (6-16). This equation is equivalent to the following condition on $\mathcal{L}^{jk\ell}_{(\pm)}$:

$$\partial_\ell \mathcal{L}^{jk\ell}_{(\pm)} = \frac{1}{c}\left(T^{jk} - T^{kj}\right)_{(\pm)} = \frac{1}{c}\left(t^{jk} - t^{kj}\right)_{(\pm)} \qquad (6\text{-}33)$$

The definitions for the total angular momentum density $M^{jk\ell}$ for the entire magnetogas-dynamical system including fluids and fields is analogous to (6-27):

$$M^{jk\ell} \equiv \mathcal{L}^{jk\ell}_{(+)} + \mathcal{L}^{jk\ell}_{(-)} + \frac{1}{c}\left(x^j T^{k\ell} - x^k T^{j\ell}\right) \qquad (6\text{-}34)$$

The condition for conservation of total angular momentum is

$$\partial_\ell M^{jk\ell} = 0 \qquad (6\text{-}35)$$

which is automatically satisfied if condition (6-33) is satisfied.

The integral form of (6-35) is

$$\mathcal{M}^{jk} \equiv \int_{V_3} M^{jk0}\, dV_3 = \text{constant} \qquad (6\text{-}36)$$

or

$$\vec{\mathcal{M}} = \int_{V_3} \vec{M}\, dV_3 = \int_{V_3} \left[\vec{\mathcal{L}}_{(+)} + \vec{\mathcal{L}}_{(-)} + \vec{r} \times \vec{\mathcal{G}}\right] dV_3 = \text{constant} \qquad (6\text{-}37)$$

where

$$\vec{\mathcal{M}} = \left(\mathcal{M}_x, \mathcal{M}_y, \mathcal{M}_z\right) \equiv \left(\mathcal{M}^{23}, \mathcal{M}^{31}, \mathcal{M}^{12}\right) \qquad (6\text{-}38)$$

It is possible to define a 4-vector total angular momentum Σ^j in terms of the antisymmetric tensor $\mathcal{M}^{jk}$ and the 4-velocity v^j of the total system as follows:

$$\Sigma^j \equiv -\frac{1}{2}\,\varepsilon^{jk\ell n}\, v_k\, \mathcal{M}_{\ell n} \tag{6-39}$$

In the rest-frame of the total system $(\vec{V}=0)$, we have

$$\left.\begin{aligned} \mathring{\Sigma}^0 &= 0 \\ (\mathring{\Sigma}^1, \mathring{\Sigma}^2, \mathring{\Sigma}^3) &= \vec{\mathring{\mathcal{M}}} \end{aligned}\right\} \quad \text{for } \vec{V}=0 \tag{6-40}$$

VII SPINOR DESCRIPTION OF A PARTICLE

Tetrapod Description

The particles with which we are dealing, electrons and protons, have an internal structure characterized by a symmetry axis, as evidenced by the existence of particle spin and magnetic dipole moment. They also possess a de Broglie phase, which must be regarded as an intrinsic property of the particle. Thus, for a complete description of a particle, in addition to its velocity, we must specify the orientation of its symmetry axis and the magnitude of its de Broglie phase. As illustrated in Figure 2, this can be accomplished by means of a set of four orthonormal 4-vectors - one time-like and three space-like vectors. This set of 4-vectors is called the particle tetrapod.

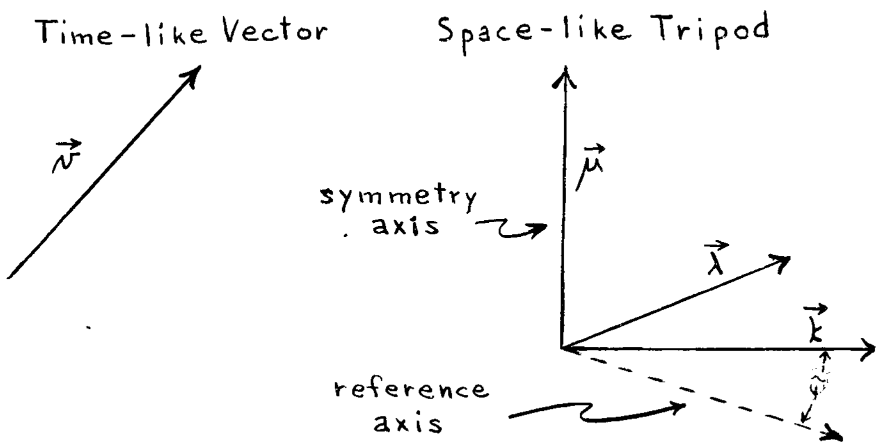

Figure 2 - Tetrapod Representation of a Particle

The time-like vector of the tetrapod is the 4-velocity u^j (designated in the figure by the 3-velocity $\vec{v}$). One of the three space-like vectors $\vec{\mu}$ is used to specify the symmetry axis of the particle. The remaining degree of freedom, the angle $\tilde{\psi}$ needed to specify the orientation of the other two space-like vectors $\vec{k}$ and $\vec{\lambda}$ about the axis $\vec{\mu}$, is measured between $\vec{k}$ and a fixed reference axis in the observer's frame of reference. The three 3-vectors $\vec{k}, \vec{\lambda}$, and $\vec{\mu}$ are, of course, the space-like parts of three 4-vectors, k^j, λ^j, and μ^j. In the particle rest-frame, the time-like components of these vectors are all zero.

We postulate that the angle $\tilde{\psi}$ is, to within a proportionality constant, the de Broglie phase Φ, and we postulate the proportionality constant to be $\hbar/2$ where $\hbar = 1.0542 \times 10^{-27}$ erg-sec is Planck's constant. (The factor $1/2$ is necessary in order for the theory to yield a particle spin angular momentum of $\hbar/2$, which is regarded as an experimental fact with which the theory must agree.) Thus we have

$$\Phi = \frac{\hbar}{2}\tilde{\psi} \tag{7-1}$$

(We shall later introduce the sign of the particle charge into this relation, so that (7-1), like all the relations until (7-8), is valid only for positively charged particles. For the moment, however, we shall ignore the question of the sign of the particle charge.)

It is with (7-1) that Planck's constant is introduced into the theory. It should be emphasized, however, that in spite of the incorporation of the de Broglie Hypothesis into the theory which results from (7-1), the theory is classical, rather than quantum-mechanical, because no quantization process is introduced at any point.

From (3-2) and (7-1) we have

$$p^j = -\partial^j \Phi = -\frac{\hbar}{2}\partial^j \tilde{\Psi} \tag{7-2}$$

In the particle rest-frame this becomes

$$\mathring{p}^\circ = \frac{\mathring{E}}{c} = \breve{m}c + \frac{q}{c}\mathring{A}^\circ = -\frac{\hbar}{2c}\frac{d\tilde{\Psi}}{d\tau} \tag{7-3a}$$

$$\mathring{\vec{p}} = \frac{q}{c}\mathring{\vec{A}} = \frac{\hbar}{2}\mathring{\vec{\nabla}}\tilde{\Psi} \tag{7-3b}$$

where $\mathring{E}$ is the particle rest-energy, $\mathring{\vec{A}}$ is the space-like part of the 4-vector potential, and $\mathring{A}^\circ$ is the time-like part which, by means of a proper choice of gauge, could be made to vanish. Thus we see that the angular velocity of rotation of $\mathring{\vec{k}}$ and $\mathring{\vec{\lambda}}$ about $\mathring{\vec{\mu}}$ in the particle rest-frame determines the particle rest-energy, and the spatial rate of change in the rest-frame, when we go from one particle to another, determines the particle momentum in the rest-frame which is just $\frac{e}{c}\mathring{\vec{A}}$. Since a knowledge of the particle rest-energy at all points of space amounts to a knowledge of the function $\check{m}$ which determines the inertial properties of the particle as well as its scalar potentials, we see that a knowledge of $\tilde{\Psi}$ at every point of space-time contains all the information necessary for a complete dynamical description of the particle.

Unfortunately, the function $\tilde{\Psi}$ is rather difficult to work with. In the case of a distribution of particles all at rest with respect to the observer, and having their $\mathring{\vec{\mu}}$ axes all aligned, $\tilde{\Psi}$ is easily determined. In the general case, however, $\tilde{\Psi}$ is one of three Euler angles that describe the orientation of the particle tripod in its own rest-frame, which in general is not the observer's frame. This matter is discussed at greater length in Appendix B. For present purposes, however, it is sufficient merely to appreciate the desirability of introducing a function having the essential properties of $\tilde{\Psi}$, but simpler to define and work with. From (7-2) we see that it is really the gradient of $\tilde{\Psi}$, rather than $\tilde{\Psi}$ itself, that interests us, so it will suffice to find a simply defined 4-vector ω^j that approximates the gradient of $\tilde{\Psi}$. (The choice we make for ω^j will actually approximate $-\frac{1}{2}\partial^j\tilde{\Psi}$.) We define ω^j as follows:

$$\omega^j \equiv \frac{1}{2}\lambda_{\ell}\partial^j k^\ell \qquad (7\text{-}4)$$

In order to see that

$$\omega^j \approx -\tfrac{1}{2}\,\partial^j\tilde{\varphi} \qquad (7\text{-}5)$$

we note that in the particular rest-frame, $\mathring{\omega}^j$ can be written

$$\mathring{\omega}^j = -\tfrac{1}{2}\,\mathring{\vec{\lambda}}\cdot(\mathring{\partial}^j\mathring{\vec{k}}) \qquad (7\text{-}6)$$

where $\mathring{\partial}^j$ is the gradient operator in the particle rest-frames. Now if one pictures two neighboring tripods whose orientations are nearly the same (or a single tripod whose orientation is changing with time), it is clear that the contribution to (7-6) resulting from a difference (or a time change) in the orientation of $\mathring{\vec{\mu}}$ is very small compared with the contribution resulting from the angular displacement of $\mathring{\vec{k}}$ and $\mathring{\vec{\lambda}}$ about $\mathring{\vec{\mu}}$. The fact that one of the two tripods we are comparing may not be exactly at rest, but rather have a small velocity, makes only a very small difference in the right side of (7-6). Thus, the value of $\mathring{\omega}^j$ is very close to what we would find in the case for which all the tripods are at rest and have their $\mathring{\vec{\mu}}$ axes aligned. This then is the basis for the approximate equality given in (7-5).

It should be emphasized, however, that this approximate equality has been introduced only to give an intuitive feeling for the physical significance of ω^j, but no use of (7-5) will be made in the development of the formalism to follow.

We conclude this discussion of the physical significance of ω^j by noting that its time-like component is essentially the de Broglie frequency of the particle, and its space-like part the de Broglie wave-vector.

Rewriting (7-3a), we have

$$\frac{d\tilde{\Psi}}{d\tau} = -\frac{2\overset{\circ}{E}}{\hbar} \tag{7-7}$$

Since the particle rest-energy $\overset{\circ}{E}$ must be positive, we see from Figure 2 that the direction of rotation of $\vec{\overset{\circ}{k}}$ about $\vec{\overset{\circ}{\mu}}$ is given by the <u>left</u>-hand screw rule. ($\vec{\overset{\circ}{k}}, \vec{\overset{\circ}{\lambda}}$, and $\vec{\overset{\circ}{\mu}}$, in that order, constitute a right-hand system of axes.)

Now let us identify the direction of rotation of $\vec{\overset{\circ}{k}}$ about $\vec{\overset{\circ}{\mu}}$ with the sign σ ($\sigma = \pm 1$) of the electric charge of the particle. Thus our discussion has so far concerned itself only with a positively charged particle. In the case of a negatively charged particle, we assert that the direction of rotation of $\vec{\overset{\circ}{k}}$ about $\vec{\overset{\circ}{\mu}}$ is given by the <u>right</u>-hand screw rule. Thus, if the (+) and (-) subscripts or superscripts refer respectively to positively or negatively charged particles, the foregoing relations are generalized as follows:

$$\Phi_{(\pm)} = \pm \frac{\hbar}{2} \tilde{\varphi}_{(\pm)} \tag{7-8}$$

$$p^{j}_{(\pm)} = -\partial^{j} \Phi_{(\pm)} = \mp \frac{\hbar}{2} \partial^{j} \tilde{\varphi}_{(\pm)} \tag{7-9}$$

$$\omega^{j}_{(\pm)} = \frac{1}{2} \lambda^{(\pm)}_{\ell} \partial^{j} k^{\ell}_{(\pm)} \tag{7-10}$$

$$\omega^{j}_{(\pm)} \approx -\frac{1}{2} \partial^{j} \tilde{\varphi}_{(\pm)} \tag{7-11}$$

$$\left(\frac{d\tilde{\varphi}_{(\pm)}}{dT}\right)_{(\pm)} = \mp \frac{2\mathring{E}_{(\pm)}}{\hbar} \tag{7-12}$$

From (7-9) and (7-11) we have

$$p^{j}_{(\pm)} \approx \pm \hbar \omega^{j}_{(\pm)} \tag{7-13}$$

and

$$\mathring{\omega}^{\circ}_{(\pm)} \approx \pm \frac{\mathring{p}^{\circ}_{(\pm)}}{\hbar} = \pm \frac{\mathring{E}_{(\pm)}}{\hbar c} \tag{7-14}$$

Thus, in contradistinction to $\overset{\circ}{p}{}^{\circ}_{(\pm)}$ which is always positive, $\overset{\circ}{\omega}{}^{\circ}_{(\pm)}$ has the same sign as the electric charge of the particle.

We now complete our tetrapod description by normalizing the four 4-vectors to $\rho_{(\pm)}$, the invariant particle density, rather than to unity. In this way we have packed all the information we need for a complete description of a charged fluid into the particle tetrapod. Altogether this description involves seven degrees of freedom: three for $\vec{v}$, two for $\overset{\circ}{\vec{\mu}}$, and one apiece for $\tilde{\psi}$ and ρ. Since each of the 4-vectors has four elements, however, our tetrapod involves a total of sixteen elements. But it has only seven degrees of freedom. Thus the elements of the tetrapod do not represent the "normal coordinates" of the problem. This fact makes the tetrapod difficult to use in any formalism.

We shall now show that it is possible to express the four 4-vectors of the tetrapod as bilinear forms of the complex elements of two spinors, each having two components. Thus the two spinors together have eight degrees of freedom. One of these degrees of freedom, the phase common to the two spinors, will be used to specify the sign of the particle charge. (It is shown in Appendix C that this identification, together with the spinor equations of motion, causes the direction of rotation of $\overset{\circ}{\vec{k}}$ and $\overset{\circ}{\vec{\lambda}}$ about $\overset{\circ}{\vec{\mu}}$ to be dependent on the sign of the particle charge, as indicated by (7-14).) If we regard the sign of the particle charge as a degree of freedom, then we can assert that the two spinors together are completely specified by eight real functions and, since the charged fluid has eight degrees of freedom, the two spinors constitute the "normal mode description" of the problem.

Spinor Description

As a preliminary to introducing the spinors that will be used to describe the charged fluids, we must first introduce the irreducible form of a 4-vector. For reasons that will shortly become clear, the superscripts and subscripts used to label the elements of a 4-vector in irreducible form will not be the usual ones ($j = 0, 1, 2, 3$). Rather, two indices α and $\bar{\beta}$, where $\alpha = 1, 2$ and $\bar{\beta} = \bar{1}, \bar{2}$, will be used. The overhead bar in $\bar{\beta}$ indicates complex conjugation, and we shall see shortly how the need for such a notation enters the formalism. First, we must define the irreducible vector elements in terms of the ordinary (reducible) elements. Using the 4-vector $(\rho u^{j})_{(\pm)}$ as an example, and omitting for the time-being the $(\pm)$ subscript, we have

$$\begin{aligned}
\rho u^{1\bar{1}} &= \frac{\rho}{\sqrt{2}}(u^0 + u^3) = \frac{\rho}{\sqrt{2}} e^{\nu} \\
\rho u^{2\bar{2}} &= \frac{\rho}{\sqrt{2}}(u^0 - u^3) = \frac{\rho}{\sqrt{2}} e^{-\nu} \\
\rho u^{1\bar{2}} &= \frac{\rho}{\sqrt{2}}(u^1 - iu^2) = \frac{\rho}{\sqrt{2}} \sinh\nu \cos\hat{\theta}\, e^{-i\hat{\varphi}} \\
\rho u^{2\bar{1}} &= \frac{\rho}{\sqrt{2}}(u^1 + iu^2) = \frac{\rho}{\sqrt{2}} \sinh\nu \cos\hat{\theta}\, e^{i\hat{\varphi}}
\end{aligned} \tag{7-15}$$

where tank

$$\nu = \tanh^{-1}(v/c) \tag{7-16}$$

and $\hat{\theta}$ and $\hat{\varphi}$ are respectively the polar and azimuthal angles of a system of spherical coordinates referred to the z axis as pole.

It is obvious that, for a rotation of the coordinate system through an angle $\Delta\hat{\varphi}$ around the z axis, we have

$$\begin{aligned}
(\rho u^{1\bar{2}})' &= e^{-i\Delta\hat{\varphi}}(\rho u^{1\bar{2}}) \\
(\rho u^{2\bar{1}})' &= e^{i\Delta\hat{\varphi}}(\rho u^{2\bar{1}})
\end{aligned} \tag{7-17}$$

where the primes indicate the values of the elements after the rotation. Under this same symmetry operation, the components ρu^1 and ρu^2 are not, however, simply multiplied by a constant as in (7-17). Rather they become shuffled, i.e., the expression for $(\rho u^1)'$ involves <u>both</u> ρu^1 and ρu^2, and similarly for $(\rho u^2)'$. Thus $\rho u^{1\bar{2}}$ and

$\rho u^{2\bar{1}}$ are irreducible representations of the symmetry group of rotations about the $\mathbf{z}$ axis, whereas ρu^1 and ρu^2 are not. Similarly, $\rho u^{1\bar{1}}$ and $\rho u^{2\bar{2}}$ are irreducible representations of the symmetry group consisting of all Lorentz transformations in the $\mathbf{z}$ direction, whereas ρu^0 and ρu^3 are not.

It is well-known that the equations of physics assume their simplest form when they are expressed in terms of irreducible quantities. For this reason, the components $\rho u^{\alpha\bar{\beta}}$ on the left side of (7-15) are to be regarded as more fundamental than the usual components ρu^j.

We now introduce the two fundamental spinors ξ^α and χ^α by means of the following relation:

$$\rho \begin{pmatrix} u^{1\bar{1}} & u^{1\bar{2}} \\ u^{2\bar{1}} & u^{2\bar{2}} \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \begin{pmatrix} \xi^{\bar{1}} & \xi^{\bar{2}} \end{pmatrix} + \begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix} \begin{pmatrix} \chi^{\bar{1}} & \chi^{\bar{2}} \end{pmatrix} \qquad (7\text{-}18)$$

where

$$\xi^{\bar{\alpha}} = \overline{\xi^{\alpha}} \;;\; \chi^{\bar{\alpha}} = \overline{\chi^{\alpha}} \qquad (\alpha = 1, 2) \qquad (7\text{-}19)$$

On the left side of (7-18) we have arranged the elements of $\rho u^{\alpha\bar{\beta}}$ in the form of a 2×2 matrix, and on the right side we have the sum of two outer products in which the pre-factor of each product is a column matrix and the post-factor is the complex conjugate of this column matrix written in row form, i.e., its Hermitean adjoint.

It is convenient, as indicated in (7-19), to designate complex conjugation of a spinor element by means of a bar over only the index(*), rather than over the entire symbol for the element. Thus, the indices $\alpha = 1, 2$ and $\bar{\beta} = \bar{1}, \bar{2}$ are regarded as entirely independent; i.e., it is not necessary in the case of $\rho\mathcal{U}^{\alpha\bar{\beta}}$ that, for example, $\beta = \bar{1}$ when $\alpha = 1$. A 4-vector in irreducible form is characterized by one of each type of index. It can be shown that an antisymmetric world tensor of second rank is characterized by either two barred or two unbarred indices. A symmetric world tensor of second rank is characterized by four indices, two of each kind. In keeping with the customary notation, vector and tensor indices ranging over the values 0, 1, 2, 3 will be designated by lower case Roman letters, whereas spinor indices will always be designated by lower case Greek letters.

(*) A commonly used spinor notation employs a dot, rather than a bar, over the index. This notation is appropriate when the spinor so designated, e.g. $\rho^{\dot{\alpha}}$, transforms like $\overline{\rho^{\alpha}}$, but is not equal to $\overline{\rho^{\alpha}}$, i.e., it differs from $\overline{\rho^{\alpha}}$ by a factor that is a scalar invariant. In order to emphasize that $\rho^{\bar{\alpha}}$ not only transforms like $\overline{\rho^{\alpha}}$, but is exactly equal to it, we use a bar over the index rather than a dot.

In order to explain why two outer products, rather than just one, are needed on the right side of (7-18), we note that, although the spinor ξ^{α} (or χ^{α}) itself has four degrees of freedom, one of these, namely the phase factor that is common to both ξ^{1} and ξ^{2}, is suppressed when we carry out the outer multiplication between ξ^{α} and its conjugate $\xi^{\bar{\alpha}}$. Thus ξ^{α} by itself can be used only to represent a 4-vector with no more than three degrees of freedom, i.e., a null vector. To represent an arbitrary time-like or space-like 4-vector, two spinors are necessary.

It is in the sense of the relation (7-18) that we may think of spinors as the "square roots" of 4-vectors. In the same way that, when we take the square root of a real number, we get an extra degree of freedom, namely the sign of the square root, we find that the "square root" of a 4-vector has extra degrees of freedom. This is obviously the case because the 4-vector ρu^{j} (or $\rho u^{\alpha\bar{\beta}}$) has four degrees of freedom, whereas the two spinors ξ^{α} and χ^{α} together have a total of eight degrees of freedom.

The physical significance of these extra degrees of freedom becomes clear when we note that, having the two spinors ξ^{α} and χ^{α}, it is possible to generate the complete particle tetrapod as follows:

$$
\begin{aligned}
\rho u^{\alpha\bar{\beta}} &= \xi^{\alpha}\xi^{\bar{\beta}} + \chi^{\alpha}\chi^{\bar{\beta}} \\
\rho \mu^{\alpha\bar{\beta}} &= \xi^{\alpha}\xi^{\bar{\beta}} - \chi^{\alpha}\chi^{\bar{\beta}} \\
\rho k^{\alpha\bar{\beta}} &= \xi^{\alpha}\chi^{\bar{\beta}} + \chi^{\alpha}\xi^{\bar{\beta}} \\
\rho \lambda^{\alpha\bar{\beta}} &= -i(\xi^{\alpha}\chi^{\bar{\beta}} - \chi^{\alpha}\xi^{\bar{\beta}})
\end{aligned}
\tag{7-20}
$$

Thus the extra degrees of freedom are to be associated with the particle tripod (k^j, λ^j, μ^j), and hence are the degrees of freedom specifying the orientation of the spin axis, and the de Broglie phase. The phase angle common to ξ^α and χ^α, which does not make itself directly felt in the generation of the tetrapod, will be used to specify the sign of the particle charge. (In Appendix C it is shown that this identification has the consequence that, for solutions of the spinors equations of motion, the direction of rotation of $\vec{\mathring{k}}$ and $\vec{\mathring{\lambda}}$ about $\vec{\mathring{\mu}}$ is dependent on the sign of the particle charge. It is only in this indirect way that the phase common to ξ^α and χ^α makes itself felt—via the spinor equations of motion—in the particle tetrapod.) Finally, we note that it can be shown that the tetrapod defined by (7-20) automatically satisfies the necessary orthonormality requirements.

Equations (7-20) provide the bridge between the spinors ξ^α and χ^α, which we regard as constituting the most fundamental description of a fluid of charged particles, and the particle tetrapod in irreducible form. To complete the formalism we need a bridge that connects the spinors directly with the 4-vectors of the tetrapod expressed in the usual form, rather than with the irreducible form as in (7-20). To accomplish this, we first rewrite (7-15) in matrix form as follows:

$$\rho\begin{pmatrix} u^{1\bar{1}} & u^{1\bar{2}} \\ u^{2\bar{1}} & u^{2\bar{2}} \end{pmatrix} = \frac{\rho}{\sqrt{2}}\begin{pmatrix} [u^0+u^3] & [u^1-iu^2] \\ [u^1+iu^2] & [u^0-u^3] \end{pmatrix}$$

$$= \frac{\rho}{\sqrt{2}}\left[u^0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + u^1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + u^2\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + u^3\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \tag{7-21}$$

We introduce the constant matrices $\sigma_j^{\alpha\bar{\beta}}$ defined as follows:

$$\sigma_0^{\alpha\bar{\beta}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1^{\alpha\bar{\beta}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2^{\alpha\bar{\beta}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3^{\alpha\bar{\beta}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7-22}$$

where, following the usual convention, the first index α indicates the row of the matrix, and the second index $\bar{\beta}$ indicates the column. These matrices, of course, are just the Pauli spin matrices. Using these matrices, (7-15) or (7-21) can be written in the following condensed form:

$$\rho u^{\alpha\bar{\beta}} = \sigma_j^{\alpha\bar{\beta}}(\rho u^j) \tag{7-23}$$

where, as always, a repeated index means contraction.

The matrices $\sigma_j^{\alpha\bar{\beta}}$ satisfy the following important relations:

$$\sigma_j^{\alpha\bar{\mu}}\,\sigma^j_{\bar{\nu}\beta} = \delta^\alpha_\beta\,\delta^{\bar{\mu}}_{\bar{\nu}} \tag{7-24a}$$

$$\sigma_j^{\alpha\bar{\beta}}\,\sigma^k_{\bar{\beta}\alpha} = \delta^k_j \tag{7-24b}$$

and

$$\begin{aligned} \overline{\sigma_j^{\alpha\bar{\beta}}} &= \sigma_j^{\beta\bar{\alpha}} \\ \overline{\sigma^j_{\bar{\alpha}\beta}} &= \sigma^j_{\bar{\beta}\alpha} \end{aligned} \tag{7-24c}$$

where δ^α_β . $\delta^{\bar{\mu}}_{\bar{\nu}}$, and δ^k_j are Kronecker delta functions. and the matrices $\sigma^j_{\bar{\alpha}\beta}$ have the same form as the matrices $\sigma_j^{\alpha\bar{\beta}}$, i.e

$$\sigma^j_{\bar{\alpha}\beta} = \begin{pmatrix} \sigma^j_{\bar{1}1} & \sigma^j_{\bar{1}2} \\ \sigma^j_{\bar{2}1} & \sigma^j_{\bar{2}2} \end{pmatrix} = \begin{pmatrix} \sigma_j^{1\bar{1}} & \sigma_j^{1\bar{2}} \\ \sigma_j^{2\bar{1}} & \sigma_j^{2\bar{2}} \end{pmatrix} = \sigma_j^{\alpha\bar{\beta}} \tag{7-25}$$

(Note that in the case of $\sigma^j_{\bar{\alpha}\beta}$ the barred index, which designates the row of the matrix, is written first whereas, in the case of $\sigma_j^{\alpha\bar{\beta}}$, it is written second and designates the column.) Using (7-24b), it is possible to invert (7-23):

$$\rho u^{j} = \sigma^{j}_{\bar{\beta}\alpha}(\rho u^{\alpha\bar{\beta}}) \tag{7-26}$$

Multiplying the equations (7-20) by $\sigma^{j}_{\bar{\beta}\alpha}$ and contracting the $\bar{\beta}$ and α, we arrive at the desired bridge between the spinors and the entire tetrapod in its usual (reducible) form:

$$\begin{aligned}
\rho u^{j} &= \sigma^{j}_{\bar{\beta}\alpha}(\xi^{\alpha}\xi^{\bar{\beta}} + \chi^{\alpha}\chi^{\bar{\beta}}) \\
\rho \mu^{j} &= \sigma^{j}_{\bar{\beta}\alpha}(\xi^{\alpha}\xi^{\bar{\beta}} - \chi^{\alpha}\chi^{\bar{\beta}}) \\
\rho k^{j} &= \sigma^{j}_{\bar{\beta}\alpha}(\xi^{\alpha}\chi^{\bar{\beta}} + \chi^{\alpha}\xi^{\bar{\beta}}) \\
\rho \lambda^{j} &= -i\sigma^{j}_{\bar{\beta}\alpha}(\xi^{\alpha}\chi^{\bar{\beta}} - \chi^{\alpha}\xi^{\bar{\beta}})
\end{aligned} \tag{7-27}$$

Thus far we have used only contravariant spinors, whose indices have been indicated by superscripts rather than subscripts. Knowing the contravariant form of a spinor, we can define the covariant form of this same spinor by means of the requirement that the contraction of the two must be an invariant. Furthermore, this invariant must be zero. Otherwise, as it can easily be shown, the invariant norm of the null vector that is the "square" of the spinor would have to be non-zero, and this cannot be. Thus we define the covariant spinors ρ_α and χ_α as follows:

$$\rho_1 = -\rho^2 \quad ; \quad \rho_2 = \rho^1$$
$$\chi_1 = -\chi^2 \quad ; \quad \chi_2 = \chi^1 \tag{7-28}$$

Thus

$$\rho_\alpha \rho^\alpha = \rho_1 \rho^1 + \rho_2 \rho^2 = -\rho^2 \rho^1 + \rho^1 \rho^2 = 0 \tag{7-29}$$

Similarly

$$\chi_\alpha \chi^\alpha = 0 \tag{7-30}$$

The contraction of ρ_α and χ^α is, however, non-zero. It follows from (7-28) that

$$\rho_\alpha \chi^\alpha = -\rho^\alpha \chi_\alpha \tag{7-31}$$

Thus raising one index in a contraction and lowering the other changes the sign of the result. This is to be contrasted with the fact that when we do the same thing with the contraction of two 4-vectors, the result is unaltered. For example,

$$p_j u^j = p^j u_j \tag{7-32}$$

For a further discussion of the algebraic aspects of spinor analysis, the reader is referred to the papers of van der Waerden (11), Laporte and Uhlenbeck (12), and Bade and Jehle (13).

By means of the relations (7-27), it is possible to write the components of $\xi^{\alpha}_{(\pm)}$ and $\chi^{\alpha}_{(\pm)}$ for each of the two charged fields in terms of the Euler angles that specify the orientation of the corresponding space-like tripod in the rest-frame of the fluid in question, and in terms of the magnitude and direction of the fluid velocity. This is carried out in Appendix B. However, for our present purposes the complete result of Appendix B is not needed. We only need to know that $\xi^{\alpha}_{(\pm)}$ and $\chi^{\alpha}_{(\pm)}$ can be expressed in terms of two normalized spinors $\hat{\xi}^{\alpha}_{(\pm)}$ and $\hat{\chi}^{\alpha}_{(\pm)}$ as follows:

$$\xi^{\alpha}_{(\pm)} = \left(\frac{\rho e^{i\epsilon}}{\sqrt{2}}\right)^{1/2}_{(\pm)} e^{-i\tilde{\psi}/2}\, \hat{\xi}^{\alpha}_{(\pm)} \tag{7-33a}$$

$$\chi^{\alpha}_{(\pm)} = \left(\frac{\rho e^{i\epsilon}}{\sqrt{2}}\right)^{1/2}_{(\pm)} e^{i\tilde{\psi}/2}\, \hat{\chi}^{\alpha}_{(\pm)} \tag{7-33b}$$

where

$$\left(\hat{\xi}_{\alpha}\,\hat{\chi}^{\alpha}\right)_{(\pm)} = 1 \tag{7-34}$$

and $\hat{\xi}^{\alpha}_{(\pm)}$ and $\hat{\chi}^{\alpha}_{(\pm)}$ are not functions of $\tilde{\psi}$, which means that they contain none of the high-frequency (of order of the de Broglie frequency) time or space oscillation associated with $\tilde{\psi}$, but only the slow variation associated with changes in fluid velocity and spin orientation. From (7-33) and (7-34), it follows that

$$\left(\xi_{\alpha}\,\chi^{\alpha}\right)_{(\pm)} = \left(\frac{\rho e^{i\epsilon}}{\sqrt{2}}\right)_{(\pm)} \tag{7-35}$$

ϵ is the phase that is common to both ξ^{α} and χ^{α}. It is obvious from (7-27) that ϵ has no direct effect on the vectors of the tetrapod. Thus ϵ is a disposable degree of freedom that we are free to use to specify the sign of the particle charge. Thus we impose the following condition on $\epsilon_{(\pm)}$:

$$\cos\epsilon_{(\pm)} = \pm 1 \tag{7-36a}$$

which is equivalent to

$$e^{i\epsilon_{(\pm)}} = \pm 1 \tag{7-36b}$$

Thus (7-35) becomes

$$\left(\varsigma_\alpha \chi^\alpha\right)_{(\pm)} = \pm \frac{\rho_{(\pm)}}{\sqrt{2}} \tag{7-37}$$

The condition (7-36) is fulfilled if we impose the following constraint on our spinors:

$$\mathcal{I}\left(\varsigma_\alpha \chi^\alpha\right)_{(\pm)} = 0 \tag{7-38}$$

where $\mathcal{I}$ designates the imaginary part of the argument. The only constraint on the real part of $\left(\varsigma_\alpha \chi^\alpha\right)_{(\pm)}$ is that it have the same sign at every point of space-time.

From (7-27) and (7-35), we have the following expression for the 4-velocity u^j as a function of the spinors:

$$u^j_{(\pm)} = \left\{\left[2(\varsigma_{\bar\mu}\chi^{\bar\mu})(\varsigma_\nu\chi^\nu)\right]^{-1/2}\left[\sigma^j_{\bar\alpha\beta}\left(\varsigma^\beta\varsigma^{\bar\alpha} + \chi^\beta\chi^{\bar\alpha}\right)\right]\right\}_{(\pm)} \tag{7-39}$$

This identity is valid regardless of whether or not the constraint (7-38) is satisfied. From (7-29) - (7-31), (7-24), and (7-39) it follows automatically that

$$\left(u^j u_j\right)_{(\pm)} = 1 \tag{7-40}$$

Thus the $u^j_{(\pm)}$ that we calculate from the spinors is automatically normalized, and so when we work with spinors there is no need for the troublesome nonlinear side condition that we encountered in (6-1c) or (6-6c). This is true even when the constraint (7-38) is dropped which, as we shall later show, is permissible in problems on a laboratory or astronomical scale.

Now let us estimate the magnitude of the error involved in the approximate equalities (7-11) and (7-13). To do this, we substitute the expressions for k^j and λ^j given in (7-27) into the definition of ω^j given in (7-10) which yields

$$\omega^j_{(\pm)} = -\tfrac{1}{2}\partial^j\tilde{\varphi}_{(\pm)} + \Upsilon^j_{(\pm)} \tag{7-41}$$

where

$$\Upsilon^j_{(\pm)} = \mathcal{A}(\hat{\chi}^\alpha \partial^j \hat{\rho}_\alpha)_{(\pm)} \tag{7-42}$$

Comparing (7-41) with (7-11), we see that $\Upsilon^j_{(\pm)}$ is the amount by which (7-11) fails of being an exact equality. It is easy to estimate the magnitude of $\Upsilon^j_{(\pm)}$. Let L be a length characterizing the size of the system in which we are interested. Now we have already noted that because $\hat{\rho}^\alpha_{(\pm)}$ and $\hat{\chi}^\alpha_{(\pm)}$ are not functions of $\tilde{\varphi}$, which contains all the fast space-time oscillation associated with the de Broglie phase, they are functions only of particle velocity and spin orientation. We shall limit ourselves to solutions in which no fast precession of the spin axis is present. For such cases it is always true that appreciable changes in $\hat{\rho}^{\alpha}_{(\pm)}$ and $\hat{\chi}^{\alpha}_{(\pm)}$ occur only over distances comparable with L and in times greater than L/c. Thus it is true that, in such cases,

$$\Upsilon^{j}_{(\pm)} \sim \frac{1}{L} \tag{7-43}$$

where the symbol $\sim$ is to be read "is of the order of".

Substituting (7-41) into (7-9), we have

$$p^{j}_{(\pm)} = -\partial^{j}\Phi_{(\pm)} = \pm\hbar(\omega^{j} - \Upsilon^{j})_{(\pm)} \tag{7-44}$$

which is the exact version of (7-13).

We shall now estimate the importance of $\Upsilon^{j}_{(\pm)}$ in (7-41) and (7-44). In making these estimates we shall, to simplify the notation, drop the ($\pm$) subscript with the understanding that the results apply equally well to either charged fluid. First we note that, since the effect of Υ^{j} in (7-44) is (as we shall see) small, we have

$$|\hbar\omega^{0}| \sim mc \tag{7-45a}$$

$$|\hbar\vec{\omega}| \sim |\vec{p}| \sim |m\vec{v}| = mv \tag{7-45b}$$

where, for order of magnitude estimates, the contribution $\frac{q}{c}A^j$ to p^j has been neglected. (The magnitude of A^j is, in any case, uncertain because of the arbitrariness in the choice of gauge.) From (7-43) and (7-45) we have

$$\frac{|\gamma^0|}{|\omega^0|} \sim \frac{\hbar L^{-1}}{mc} = \frac{\lambda\!\!\!^{-}}{L} \tag{7-46a}$$

$$\frac{|\vec{\gamma}|}{|\vec{\omega}|} \sim \frac{\hbar L^{-1}}{mv} = \left(\frac{\lambda\!\!\!^{-}}{L}\right)\left(\frac{c}{v}\right) \tag{7-46b}$$

where

$$\lambda\!\!\!^{-} = \frac{\hbar}{mc} \tag{7-47}$$

is the Compton wavelength of the particle. Let $O(n)$ designate "a function of the order of magnitude of the number n". Using (7-46), we can write (7-41) and (7-44) as follows:

$$\frac{\partial\tilde{\psi}}{\partial t} = -2c\omega^0\left[1 + O(\lambda\!\!\!^{-}/L)\right] \tag{7-48a}$$

$$\vec{\nabla}\tilde{\varphi} = 2\vec{\omega}\left[1 + O\left(\frac{\lambda c}{L v}\right)\right] \tag{7-48b}$$

$$p^0 = \sigma\hbar\,\omega^0\left[1 + O(\lambda/L)\right] \tag{7-49a}$$

$$\vec{p} = \sigma\hbar\,\vec{\omega}\left[1 + O\left(\frac{\lambda c}{L v}\right)\right] \tag{7-49b}$$

where $\sigma = \pm 1$ is the sign of the particle charge.

To estimate (λ/L) and $(\lambda/L)(c/v)$ we use $\lambda \sim 10^{-11}$ cm, which is appropriate for an electron; $L \sim 10^{11}$ cm which is the order of magnitude of the sun's radius; and $v \sim 10^5$ cm/sec, which is the order of magnitude of the rotational velocity at the surface of the sun. For these values we have

$$\frac{\lambda}{L} \sim 10^{-24} \tag{7-50a}$$

$$\frac{\lambda c}{L v} \sim 10^{-19} \tag{7-50b}$$

which are very small indeed. We note that, in the neighborhood of a stagnation point where $v = 0$, the contribution $\frac{q}{c}A^j$ to p^j, which we neglected in (7-45), becomes important. In such a region we should replace ν in (7-45) - (7-50) by $\left|\frac{q}{mc}\vec{A}\right|$. This guarantees that the ratio (7-46b) remains small, even when $\nu = 0$, since in general $\vec{A} \neq 0$ at such points. (In fact, we can guarantee that this is the case by an appropriate choice of gauge.)

The sole purpose of these estimates has been to demonstrate that, at least for the type of problem in which we are interested, the vector ω^j that we introduced to replace $-\frac{1}{2}\partial^j\tilde{\varphi}$, which was difficult to work with in calculations, is indeed very nearly equal to $-\frac{1}{2}\partial^j\tilde{\varphi}$. (7-48) also provides the quantitative justification for calling ω^0 the de Broglie frequency and $\vec{\omega}$ the de Broglie wave-vector. When we recall that, according to (3-2), p^j must be the gradient of a scalar, we note from (7-48) and (7-49) that this is also very nearly the case for ω^j.

All of the relations presented in this section have followed directly from the definitions that provide the link between the spinors ξ^α, χ^α and the kinetic and dynamical properties described by the particle tetrapod. It is evident that any spinor equation of motion that determines the space-time behavior of ξ^α and χ^α for given electromagnetic, gravitational, and thermal fields will indirectly specify the equations of motion of the fluid quantities described by the tetrapod, in particular the equations of motion of ρ and u^j. If it should turn out that the equations so determined are just the systems (6-1) or (6-6), then the postulated spinor equation of motion for ξ^α and χ^α is the desired spinor alternative to Euler's equation.

In the next section, we shall postulate a certain spinor equation of motion, and then using the definitions of the present section we shall derive the equations of motion satisfied by the quantities described by the tetrapod. We shall find that the resulting equations are identical to the system (6-6) supplemented by an equation of motion for the particle spin, which is interesting but, for the purposes of magnetogas dynamics, not of primary importance. The spin-dependent potential $a^j_{(\pm)}$ in (6-6a) will be given as an explicit function of $\rho^\alpha_{(\pm)}$ and $\chi^\alpha_{(\pm)}$, and it will be shown that, for macroscopic problems, $a^j_{(\pm)}$ is completely negligible compared with the potentials A^j that are encountered in such problems.

Section VIII will merely summarize results. The derivation of these results is carried out in Appendix C. For a detailed understanding of Appendix C, some prior familiarity with spinor analysis would be helpful, and for this the reader is referred to references (11), (12), and (13).

As a necessary preliminary to discussing the spinor equation of motion, we note that the 4-gradient operator ∂_j can also be written with spinor indices. Thus, analogous to (7-21), we have

$$\begin{pmatrix} \partial_{\dot{1}1} & \partial_{\dot{1}2} \\ \partial_{\dot{2}1} & \partial_{\dot{2}2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \left[\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right] & \left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right] \\ \left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right] & \left[\frac{1}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right] \end{pmatrix} \qquad (7\text{-}51)$$

which can also be written

$$\partial_{\bar{\alpha}\beta} = \sigma^{j}_{\bar{\alpha}\beta}\, \partial_j \tag{7-52}$$

whose inverse is

$$\partial_j = \sigma_j^{\ \beta\bar{\alpha}}\, \partial_{\bar{\alpha}\beta} \tag{7-53}$$

The following operator identities are also of interest:

$$\partial^{\alpha\bar{\lambda}}\, \partial_{\bar{\lambda}\beta} = \partial_{\bar{\lambda}\beta}\, \partial^{\alpha\bar{\lambda}} = \tfrac{1}{2}\delta^{\alpha}_{\beta}\, \partial_{\ell}\, \partial^{\ell} = \tfrac{1}{2}\delta^{\alpha}_{\beta}\, \Box \tag{7-54}$$

$$\sigma_j^{\ \beta\bar{\nu}}\, \sigma^{k}_{\bar{\nu}\alpha}\, \partial^{\alpha\bar{\lambda}}\, \partial_{\bar{\lambda}\beta} = \tfrac{1}{2}\delta^{k}_{j}\, \partial_{\ell}\, \partial^{\ell} = \tfrac{1}{2}\delta^{k}_{j}\, \Box \tag{7-55}$$

From (7-54) it follows that

$$\Box = \partial_{\ell}\, \partial^{\ell} = \partial_{\bar{\alpha}\beta}\, \partial^{\beta\bar{\alpha}} \tag{7-56}$$

In the next section we shall work with the irreducible covariant form $A_{\bar{\alpha}\beta}$ of the 4-vector potential where

$$\begin{pmatrix} A_{\bar{1}1} & A_{\bar{1}2} \\ A_{\bar{2}1} & A_{\bar{2}2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} [A_0 + A_3] & [A_1 - iA_2] \\ [A_1 + iA_2] & [A_0 - A_3] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} [A^0 - A_z] & [-A_x + iA_y] \\ [-A_x - iA_y] & [A^0 + A_z] \end{pmatrix} \tag{7-57}$$

where we have made use of the fact that it is the contravariant form A^j, rather than the covariant form A_j, that is regarded as the generalization to four dimensions of the 3-vector $\vec{A} = (A_x, A_y, A_z)$.

The following identity is useful for evaluating F^{jk} from $A_{\bar{\nu}\beta}$:

$$2 g^{jl} \sigma_l^{\beta\bar{\lambda}} \sigma_{\bar{\lambda}\alpha}^{k} (\partial^{\alpha\bar{\nu}} A_{\bar{\nu}\beta}) = g^{jk}(\partial_l A^l) - (\partial^j A^k - \partial^k A^j) - i\varepsilon^{jkln} \partial_l A_n \qquad (7\text{-}58)$$

$$= g^{jk}(\partial_l A^l) - F^{jk} - i\hat{F}^{jk}$$

where we have used (4-3) and (4-5).

VIII SPINOR EQUATIONS OF MOTION

Spinor Equations

We postulate that the spinor equations of motion have the following form:

$$i\hbar\,\partial_{\bar{\alpha}\beta}\,\xi^{\beta}_{(\pm)} = \frac{\breve{m}_{(\pm)}c}{\sqrt{2}}\,\chi^{(\pm)}_{\bar{\alpha}} - \frac{q}{c}A_{\bar{\alpha}\beta}\,\xi^{\beta}_{(\pm)} \tag{8-1a}$$

$$i\hbar\,\partial_{\bar{\alpha}\beta}\,\chi^{\beta}_{(\pm)} = \frac{\breve{m}_{(\pm)}c}{\sqrt{2}}\,\xi^{(\pm)}_{\bar{\alpha}} + \frac{q}{c}A_{\bar{\alpha}\beta}\,\chi^{\beta}_{(\pm)} \tag{8-1b}$$

where

$$\breve{m}_{(\pm)} = m_{(\pm)}\left[1 + \mathcal{G}/c^2 + h_{(\pm)}/c^2\right] \tag{8-1c}$$

We supplement these equations with the constraint (7-38):

$$\mathcal{I}(\xi_{\alpha}\chi^{\alpha})_{(\pm)} = 0 \tag{8-1d}$$

which is equivalent to requiring that

$$\cos\epsilon_{(\pm)} = \sigma \quad ; \quad \sigma = \pm 1 \tag{8-1e}$$

where σ is the sign of the particle charge of the fluid under consideration.

We note that the signs preceding q in the above spinor equations are the same for both charged fluids. The reason for this is that the sign of the particle charge is treated as part of the solution of the spinor equations (the value of the phase ϵ), rather than as a parameter in the equations.

The equations (8-1) are just the spinor form of the Dirac equation with the generalization that the mass includes contributions arising from the gravitational and thermal energy.(*) It should be emphasized, however, that, except for the incorporation of the de Broglie Hypothesis into the theory, which introduced Planck's constant as a proportionality factor relating $\Phi_{(\pm)}$ and $\tilde{\Psi}_{(\pm)}$ (cf. (7-8)), the theory developed in this paper is completely classical in spirit. In particular, no quantization process has been introduced.

It is really not surprising that a Dirac-type equation should be the first candidate for investigation in a theory like the one developed here, since this equation is the simplest first-order linear equation that incorporates a scalar field $(\breve{m}_{(\pm)})$ and a vector field $(A_{\bar{\alpha}\beta})$, and also guarantees that the observable quantities of physical interest $(\rho, u^j, \mu^j, \omega^j)_{(\pm)}$ do not contain

(*) See, for example, ref. 12, p. 1393, eq. 1. This equation does not include the factor $\sqrt{2}$ appearing in (8-1a) and (8-1b) simply because of a difference in the definition of the matrices $\sigma^j_{\bar{\alpha}\beta}$.

oscillatory terms with frequency of the order of the de Broglie frequency.

Derivation of the Euler System of Equations

Using the spinor equations of motion (8-1), and the definitions of Section VII that relate the observable quantities described by the tetrapod for the fluid to the spinors $\rho^{\alpha}_{(\pm)}$ and $\chi^{\alpha}_{(\pm)}$, it is shown in Appendix C that we can arrive at the fluid equations of motion in terms of these observable quantities. In this section, we shall merely exhibit these equations, and discuss their physical significance. The most important of these equations are the following:

$$\partial_j(\rho u^j)_{(\pm)} = 0 \tag{8-2}$$

$$p^j_{(\pm)} = -\partial^j \Phi_{(\pm)} = (Mcu^j)_{(\pm)} + \frac{\sigma q}{c}\left[A^j + a^j_{(\pm)}\right] \tag{8-3}$$

where

$$a^j_{(\pm)} = \frac{\hbar c}{q}(\pi^j - \gamma^j)_{(\pm)} \tag{8-4}$$

where $\gamma^j_{(\pm)}$ is defined by (7-42) and

$$\begin{aligned}\pi^j_{(\pm)} &= \frac{1}{2\rho}\left\{\partial_k(\rho s^{jk}) - \left[u_k\partial_\ell(\rho s^{k\ell})\right]u^j\right\}_{(\pm)} \\ &= \frac{1}{2}\left[\rho^{-1}\varepsilon^{jk\ell n}u_k\partial_\ell(\rho\mu_n) - s^{jk}u^\ell\partial_\ell u_k\right]_{(\pm)}\end{aligned} \tag{8-5}$$

The $M_{(\pm)}$ that appears in (8-3) is defined as follows:

$$M_{(\pm)} \equiv \check{m}_{(\pm)} + \tilde{m}_{(\pm)} \tag{8-6}$$

where

$$\begin{aligned} \tilde{m}_{(\pm)} &= \frac{1}{2c^2}\left[\frac{q\hbar}{2M_{(\pm)}c} \, s^{jk}_{(\pm)}\left(F_{jk}+f^{(\pm)}_{jk}\right)\right] \\ &= -\frac{1}{c^2}\left[\frac{q\hbar}{2M_{(\pm)}c} \, \mathring{\vec{\mu}}\cdot(\mathring{\vec{B}}+\mathring{\vec{b}})\right]_{(\pm)} \end{aligned} \tag{8-7}$$

where

$$f^{jk}_{(\pm)} = \partial^j a^k_{(\pm)} - \partial^k a^j_{(\pm)} \tag{8-8a}$$

$$\begin{aligned} (f^{10}, f^{20}, f^{30})_{(\pm)} &\equiv (e_x, e_y, e_z)_{(\pm)} = \vec{e}_{(\pm)} \\ -(f^{23}, f^{31}, f^{12})_{(\pm)} &\equiv (b_x, b_y, b_z)_{(\pm)} = \vec{b}_{(\pm)} \end{aligned} \tag{8-8b}$$

and the overhead "o" in (8-7) indicates the value of the 3-vector in question as seen in the particle rest-frame. (Note that, in general, $\mathring{\vec{B}}$ in the case of the proton gas is different from $\mathring{\vec{B}}$ in the case of the electron gas because the two rest-frames are different.)

Note that the definition for $M_{(\pm)}$ in (8-6) is implicit, rather than explicit, because $\tilde{m}_{(\pm)}$ involves $M_{(\pm)}$ itself (rather than $m_{(\pm)}$) in the denominator.

Comparing (8-7) with (6-1f), we make the following identification:

$$\mu_{(\pm)} = \frac{q\hbar}{2M_{(\pm)}c} \tag{8-9}$$

where $\mu_{(\pm)}$ is the magnitude of the magnetic dipole moment of the particle. (8-9) is just the dipole moment that the Dirac theory yields (g-factor equals 2), with the modification that the total mass $M_{(\pm)}$, rather than the free-particle mass $m_{(\pm)}$, appears in the denominator.

The equations (8-2) and (8-3), together with the automatic normalization of the 4-velocity $u^j_{(\pm)}$ that was demonstrated in (7-40), constitute the system of equations given in (6-6). From this system we can derive, as was shown in Section VI, the Euler system of equations (6-1). Thus we have shown that a solution of the system of spinor equations (8-1) automatically yields, via the identities of Section VII, a solution of the Euler system of equations (6-1).

Magnitude of Spin-Dependent Electromagnetic Field

Now we shall demonstrate that, for magnetogas-dynamical systems on a laboratory or astronomical scale, the effects of the spin-dependent field $f^{jk}_{(\pm)}$ are completely negligible. (Recall that $f^{jk}_{(\pm)}$ has its origin in the small contribution $\delta p^j_{(\pm)}$ to the fluid momentum density that resulted from the presence of the spin angular momentum density. Thus $f^{jk}_{(\pm)}$ is really a spin-dependent mechanical effect, although it has been treated, for reasons of intuitive clarity, as a contribution to the electromagnetic field.)

In Appendix C it is shown that, for solutions of (8-1) that yield functions for ρ, u^j, μ^j, and ω^j that change significantly only over distances of the order L and in time intervals greater than L/c, we have

$$\left| f^{jk}_{(\pm)} \right| \sim \frac{q}{\alpha L^2} \tag{8-10}$$

where α is the fine-structure constant:

$$\alpha = \frac{q^2}{\hbar c} = \frac{1}{137.04} \tag{8-11}$$

Table I summarizes the comparison between $\left| f^{jk}_{(\pm)} \right|$ and the magnitudes $\left| F^{jk} \right|$ of the electromagnetic fields encountered in the sun and in a system of laboratory dimensions. Only crude estimates for the lower limits of $\left| F^{jk} \right|$ have been given, but these suffice to indicate that in macroscopic systems the effects of $f^{jk}_{(\pm)}$ are completely negligible. The third line of Table I has been included merely to demonstrate that, if one were to attempt to use the spinor formulation of magnetogas dynamics to construct models of nuclei or elementary particles, the effects of the spin-dependent field $f^{jk}_{(\pm)}$ would be not at all negligible.

Table I

	L(cm)	$\left\| f^{jk}_{(\pm)} \right\|$ (Gauss)	$\left\| F^{jk} \right\|$ (Gauss)
Sun	10^{11}	10^{-29}	>1
Laboratory	1	10^{-7}	>1
Nucleus	10^{-13}	10^{18}	10^{16}

Effect of Variable ϵ

The side condition (8-1d) is analogous to the 4-velocity normalization condition (6-1c) in the Euler system of equations in that both conditions introduce an algebraic nonlinearity into the problem, since they are quadratic in the unknown functions. We shall now demonstrate, however, that the spinor system of equations has the great advantage that, for problems on an astronomical or laboratory scale, the nonlinear side condition (8-1d) may be ignored.

In this regard, we already noted following (7-40) that dropping the constraint (8-1d) has not effect on the automatic normalization of $u^j_{(\pm)}$. Moreover, in Appendix C it is shown that, if we ignore this constraint, no change in the continuity equation (8-2) results, and the effect on (8-3) can be represented as a spurious mass contribution $(\Delta_\epsilon m)_{(\pm)}$ and a spurious 4-vector potential $(\Delta_\epsilon A^j)_{(\pm)}$. Under the same conditions used in estimating $\left| f^{jk}_{(\pm)} \right|$, we find

$$\left| (\Delta_\epsilon A^j)_{(\pm)} \right| \sim \left| a^j_{(\pm)} \right| \left(\frac{\lambda_{(\pm)}}{L} \right) \tag{8-12}$$

where $\lambda\!\!\!^{-}_{(\pm)}$ is the Compton wavelength introduced in (7-47). Thus, for macroscopic systems, the effect of $(\Delta_\epsilon A^j)_{(\pm)}$ is even smaller than that of $a^j_{(\pm)}$, whereas, for systems of nuclear dimensions, this is obviously no longer the case.

In order to estimate the effect of the spurious mass $(\Delta_\epsilon m)_{(\pm)}$, we note that this may be regarded as arising from a spurious contribution to the specific enthalpy, which in turn may be regarded as arising from an error $(\Delta_\epsilon T)_{(\pm)}$ in the fluid temperature. In Appendix C, it is shown that

$$\left|(\Delta_\epsilon T)_{(\pm)}\right| \sim \left(\frac{m_{(\pm)} c^2}{k}\right)\left(\frac{\lambda\!\!\!^{-}_{(\pm)}}{L}\right)^2 \qquad (8\text{-}13)$$

where k is the Boltzmann constant. In Table II $|\Delta_\epsilon T_{(-)}|$ (i.e.. the error in the electron temperature) is compared with the range of temperatures encountered in the sun, laboratory, and in nuclear systems. (The case for protons leads to even smaller estimates.)

Table III

	L(cm)	$\lvert\Delta_\epsilon T_{(-)}\rvert$ ($^\circ$K)	T($^\circ$K)
Sun	10^{11}	10^{-33}	10^{3}-10^{7}
Lab	1	10^{-11}	0-10^{3}
Nucleus	10^{-13}	10^{15}	0-10^{10}

Obviously, the effect of $\Delta_\epsilon T_{(\pm)}$ is completely negligible in the first two cases, and not at all negligible in the last case.

Finally, it is shown in Appendix C that

$$|\epsilon_{(+)}| \sim \frac{\lambda_{(+)}}{L} ; \quad |\pi - \epsilon_{(-)}| \sim \frac{\lambda_{(-)}}{L} \tag{8-14}$$

Thus, neglecting the side condition (8-1d) in the case of macroscopic systems will not produce a change in the sign of $\cos \epsilon_{(\pm)}$, which we interpret as the sign of the electric charge of the particle. Another way of interpreting (8-14) is to say that, for macroscopic systems, the equations (8-1a) and (8-1b) come very close to satisfying the condition (8-1d) automatically, without anything being done to bring this about.

In summation, we conclude that, in the case of macroscopic magnetogas-dynamical systems, ignoring the side condition (8-1d) will introduce no significant error into the solution.

Stress-Energy Tensor

In Appendix C it is shown that the following equation results from the spinor equations (8-1):

$$\partial_k T^{jk}_{(\pm)} = \partial_k T^{kj}_{(\pm)} = \left[\rho \partial^j (\breve{m} c^2)\right]_{(\pm)} + F^{jk}(\sigma q \rho u_k)_{(\pm)} = \mathcal{k}^j_{(\pm)} \tag{8-15}$$

where

$$T^{jk}_{(\pm)} = \left[\rho \breve{m} c^2 u^j u^k + \frac{\sigma \hbar c}{2} u^j \partial_\ell (\rho s^{k\ell}) - \frac{\sigma \hbar c}{2} \rho s^{j\ell} \partial^k u_\ell\right] \tag{8-16}$$

and $\mathcal{k}^j_{(\pm)}$ is the 4-force density acting on the fluid.

We see in (8-15) that it makes no difference whether we contract ∂_k with $T^{jk}_{(\pm)}$ or $T^{kj}_{(\pm)}$. This is a non-trivial result, inasmuch as the tensor $T^{jk}_{(\pm)}$ is not symmetric.

Equation (8-15) justifies our regarding $T^{jk}_{(\pm)}$ as the stress-energy tensor of the fluid. Comparing (8-16) with (6-11), we see that $t^{jk}_{(\pm)}$, the spin-dependent part of the stress-energy tensor, has the following form:

$$t^{jk}_{(\pm)} = \frac{\sigma\hbar c}{2}\left[u^j\partial_\ell(\rho s^{k\ell}) - \rho s^{j\ell}\partial^k u_\ell\right]_{(\pm)} \tag{8-17}$$

<u>Angular Momentum</u>

In Appendix C it is shown that the following equation results from the system of spinor equations (8-1):

$$\partial_\ell M^{jk\ell}_{(\pm)} = \frac{1}{c}\left[x^j p^k_{(\pm)} - x^k p^j_{(\pm)}\right] \tag{8-18}$$

where

$$M^{jk\ell}_{(\pm)} = -\frac{1}{c}\frac{\sigma\hbar}{2}\,\varepsilon^{jk\ell n}(\rho\mu_n)_{(\pm)} + \frac{1}{c}\left[x^j T^{k\ell}_{(\pm)} - x^k T^{j\ell}_{(\pm)}\right] \tag{8-19}$$

Comparing (8-19) with (6-27) and (6-30), we make the following identifications:

$$s^{jk\ell}_{(\pm)} = -\frac{\sigma\hbar}{2}\,\varepsilon^{jk\ell n}(\rho\mu_n)_{(\pm)} \tag{8-20}$$

and

$$\vec{\mathcal{S}}_{(\pm)} \equiv \left(\mathcal{S}^{230}, \mathcal{S}^{310}, \mathcal{S}^{120}\right)_{(\pm)} = \frac{\sigma\hbar}{2}(\rho\vec{\mu})_{(\pm)} = \left[(\rho u^0)\left(\frac{\sigma\hbar}{2}\,\frac{\vec{\mu}}{u^0}\right)\right]_{(\pm)} \tag{8-21}$$

Thus $\frac{\sigma\hbar}{2}(\rho\vec{\mu})_{(\pm)}$ is the spin angular momentun density and, since $(\rho u^0)_{(\pm)}$ is the particle density in the observer's frame, $\left(\frac{\sigma\hbar}{2}\,\frac{\vec{\mu}}{u^0}\right)_{(\pm)}$ is the spin angular momentum of a single particle in the observer's frame.

In this regard, it is interesting to note that it follows directly from the Lorentz transformation that

$$\frac{\vec{\mu}}{u^0} = \mathring{\vec{\mu}}_{\parallel} + \frac{\mathring{\vec{\mu}}_{\perp}}{u^0} \tag{8-22a}$$

where

$$\mathring{\vec{\mu}}_{\parallel} = \left(\frac{\mathring{\vec{\mu}}\cdot\vec{v}}{v^2}\right)\vec{v} \tag{8-22b}$$

is the component of $\mathring{\vec{\mu}}$ parallel to $\vec{v}$, and

$$\mathring{\vec{\mu}}_{\perp} = \frac{(v \times \mathring{\vec{\mu}}) \times \vec{v}}{v^2} \tag{8-22c}$$

is the component of $\mathring{\vec{\mu}}$ perpendicular to $\vec{v}$. Thus

$$\lim_{\vec{v}\to c}\left(\frac{\vec{\mu}}{u^0}\right) \longrightarrow \mathring{\vec{\mu}}_{\parallel} \tag{8-23}$$

which corresponds to the well-known property of particle spin to align itself parallel or antiparallel (depending on the sign of in (8-22b)) to the particle velocity at high speeds.

Finally, it is shown in Appendix C that in the fluid rest-frame the following equation is valid:

$$\left[\frac{d\left(\frac{\sigma\hbar}{2}\overset{\circ}{\vec{\mu}}\right)}{d\tau}\right]_{(\pm)} = \left[\left(\frac{q\hbar\overset{\circ}{\vec{\mu}}}{2Mc}\right)\times\left(\overset{\circ}{\vec{B}}+\overset{\circ}{\vec{b}}\right)\right]_{(\pm)} \qquad (8\text{-}24)$$

This equation describes the precession in its own rest-frame of a particle having a spin angular momentun $\left(\frac{\sigma\hbar}{2}\overset{\circ}{\vec{\mu}}\right)_{(\pm)}$ and a magnetic moment $\left(\frac{q\hbar}{2Mc}\overset{\circ}{\vec{\mu}}\right)_{(\pm)}$ corresponding to a gyromagnetic ratio $\left(\frac{\sigma q}{Mc}\right)_{(\pm)}$ which in turn corresponds to a g-factor of 2 (referred, of course, to the total particle mass $M_{(\pm)}$ rather than the free particle mass $m_{(\pm)}$). The precession produced by $\overset{\circ}{\vec{b}}_{(\pm)}$ can be regarded as the result of a self-interaction of the electron or proton fluid with itself. In any case, Table I shows us that $|\overset{\circ}{\vec{b}}_{(\pm)}| << |\overset{\circ}{\vec{B}}|$ for macroscopic systems.

<u>Dimensionless Form of Equations</u>

Let us refer the particle masses to the electron mass m_e where

$$m_e = 9.107\times10^{-28}\,gm \qquad (8\text{-}25)$$

Thus, if $m_{(-)}$ and $m_{(+)}$ are the electron and proton masses respectively, we have

$$\acute{m}_{(-)} = \frac{m_{(-)}}{m_e} = 1 \qquad (8\text{-}26a)$$

$$\acute{m}_{(+)} = \frac{m_{(+)}}{m_e} = 1836.1 \tag{8-26b}$$

where the dimensionless masses have been indicated by an overhead stroke. We shall refer all lengths to the Compton wavelength of the electron $\bar{\lambda}_e$ where

$$\bar{\lambda}_e = \frac{\hbar}{m_e c} = 3.862 \times 10^{-11}\,\mathrm{cm} \tag{8-27}$$

Thus we have the following relations between the dimensional and dimensionless (indicated by a stroke) quantities of principal interest:

$$t = (\bar{\lambda}_e/c)\,\acute{x}^0 = 1.2882 \times 10^{-21}\,\acute{x}^0\ \mathrm{sec} \tag{8-28a}$$

$$x^j = \bar{\lambda}_e\,\acute{x}^j = 3.862 \times 10^{-11}\,\acute{x}^j\ \mathrm{cm} \tag{8-28b}$$

$$\partial_j = \bar{\lambda}_e^{-1}\,\acute{\partial}_j = 2.590 \times 10^{10}\,\acute{\partial}_j\ \mathrm{cm}^{-1} \tag{8-28c}$$

$$\Box = \bar{\lambda}_e^{-2}\,\acute{\Box} = 6.707 \times 10^{20}\,\acute{\Box}\ \mathrm{cm}^{-2} \tag{8-28d}$$

$$u^j_{(\pm)} = \acute{u}^j_{(\pm)} \tag{8-28e}$$

$$\omega^j_{(\pm)} = \bar{\lambda}_e^{-1}\,\acute{\omega}^j_{(\pm)} = 2.590 \times 10^{10}\,\acute{\omega}^j_{(\pm)}\ \mathrm{cm}^{-1} \tag{8-28f}$$

$$\rho_{(\pm)} = \bar{\lambda}_e^{-3}\,\acute{\rho}_{(\pm)} = 1.7372 \times 10^{30}\,\acute{\rho}_{(\pm)}\ \mathrm{cm}^{-3} \tag{8-28g}$$

$$\begin{aligned}(\xi^\alpha, \chi^\alpha)_{(\pm)} &= \bar{\lambda}_e^{-3/2}(\acute{\xi}^\alpha, \acute{\chi}^\alpha) \\ &= 4.168 \times 10^{15}(\acute{\xi}^\alpha, \acute{\chi}^\alpha)\ \mathrm{cm}^{-3/2}\end{aligned} \tag{8-28h}$$

$$A^j = (q/\lambda_e)\acute{A}^j = 122.434\ \acute{A}^j \text{ statvolt} \tag{8-28i}$$

$$\eta_{(\pm)} = q\,\acute{\eta}_{(\pm)} = 4.802\times 10^{-10}\,\acute{\eta}_{(\pm)} \text{ statvolt-cm} \tag{8-28j}$$

$$F^{jk} = (q/\lambda_e^2)\acute{F}^{jk} = 3.220\times 10^{11}\,\acute{F}^{jk} \text{ statvolt/cm} \tag{8-28k}$$

$$g = (\alpha c^2)\acute{g} = 6.558\times 10^{18}\,\acute{g} \text{ erg/gm} \tag{8-28l}$$

$$h_{(\pm)} = (\alpha c^2)(\acute{H}\acute{h})_{(\pm)} = 6.558\times 10^{18}(\acute{H}\acute{h})_{(\pm)} \text{ erg/gm.} \tag{8-28m}$$

Two dimensionless coupling constants will be needed: the fine-structure constant α,

$$\alpha = \frac{q^2}{\hbar c} = 7.297\times 10^{-3} = \frac{1}{137.04} \tag{8-29}$$

and the dimensionless gravitational constant $\acute{\Gamma}$,

$$\acute{\Gamma} = \Gamma\left(\frac{m_e}{q}\right)^2 = 2.398\times 10^{-43} \tag{8-30}$$

The dimensionless spinor equations (8-1) have the form

$$i\,\acute{\partial}_{\bar{\alpha}\beta}\acute{\xi}^{\beta}_{(\pm)} = \frac{\acute{m}_{(\pm)}}{\sqrt{2}}\acute{\chi}^{(\pm)}_{\bar{\alpha}} + \alpha\left\{\acute{m}_{(\pm)}\left[\acute{g} + (\acute{H}\acute{h})_{(\pm)}\right]\acute{\chi}_{\bar{\alpha}} - \left[\acute{A}_{\bar{\alpha}\beta} + \acute{\partial}_{\bar{\alpha}\beta}\acute{\eta}_{(\pm)}\right]\acute{\xi}^{\beta}_{(\pm)}\right\} \tag{8-31a}$$

$$i\,\acute{\partial}_{\bar{\alpha}\beta}\acute{\chi}^{\beta}_{(\pm)} = \frac{\acute{m}_{(\pm)}}{\sqrt{2}}\acute{\xi}^{(\pm)}_{\bar{\alpha}} + \alpha\left\{\acute{m}_{(\pm)}\left[\acute{g} + (\acute{H}\acute{h})_{(\pm)}\right]\acute{\xi}_{\bar{\alpha}} + \left[\acute{A}_{\bar{\alpha}\beta} + \acute{\partial}_{\bar{\alpha}\beta}\acute{\eta}_{(\pm)}\right]\acute{\chi}^{\beta}_{(\pm)}\right\} \tag{8-31b}$$

where $\acute{\eta}_{(-)}$ and $\acute{\eta}_{(+)}$ are dimensionless real scalar functions of the space-time coordinates that determine the gauges of the 4-vector potential that one chooses to use in the solution of the spinor equations for the electron and proton gases. There is no need that the same gauge be used in the two cases. These scalar functions, which did not appear in (8-1), have been inserted in (8-31) simply to make the arbitrariness of the gauge more explicit.

The algebraically nonlinear condition (8-1d) has not been included in (8-31) because we saw that ignoring it introduced no significant error into the solutions of macroscopic problems.

The field equations (6-8) in dimensionless form are as follows:

$$\acute{\rho}_{(\pm)} \acute{\partial}_j \acute{h}_{(\pm)} = [\gamma_{(\pm)} - 1]\, \acute{h}_{(\pm)} \acute{\partial}_j \acute{\rho}_{(\pm)} \tag{8-32a}$$

or

or $$\acute{h}_{(\pm)} = \left[(\acute{\rho}/\acute{\rho}_0)^{(\gamma-1)}\right]_{(\pm)} \tag{8-32b}$$

$$\acute{\Box} \acute{A}_{\bar{\alpha}\beta} - \acute{\partial}_{\bar{\alpha}\beta}(\acute{\partial}^{\nu\bar{\lambda}} \acute{A}_{\bar{\lambda}\nu}) = 4\pi\left[(\acute{\rho}\acute{u}_{\bar{\alpha}\beta})_{(+)} - (\acute{\rho}\acute{u}_{\bar{\alpha}\beta})_{(-)}\right] \tag{8-32c}$$

$$\acute{\Box} \acute{g} = -4\pi \acute{\Gamma}\left[(\acute{\rho}\acute{m})_{(+)} + (\acute{\rho}\acute{m})_{(-)}\right] \tag{8-32d}$$

where from (7-28) and (7-35)

$$\acute{\rho}_{(\pm)} = \sqrt{2}\left|\acute{\xi}_\alpha \acute{\chi}^\alpha\right|_{(\pm)} = \sqrt{2}\left|\acute{\xi}^1 \acute{\chi}^2 - \acute{\xi}^2 \acute{\chi}^1\right|_{(\pm)} \tag{8-32e}$$

and from (7-20)

$$(\acute{\rho}\acute{u}_{\bar{\alpha}\beta})_{(\pm)} = (\acute{\xi}_{\bar{\alpha}} \acute{\xi}_\beta + \acute{\chi}_{\bar{\alpha}} \acute{\chi}_\beta)_{(\pm)} \tag{8-32f}$$

or, using (7-28)

$$
\begin{aligned}
(\acute{\rho}\acute{u}_{\bar{1}1})_{(\pm)} &= (\acute{\xi}^{\bar{2}}\acute{\xi}^{2} + \acute{\chi}^{\bar{2}}\acute{\chi}^{2})_{(\pm)} \\
(\acute{\rho}\acute{u}_{\bar{2}2})_{(\pm)} &= (\acute{\xi}^{\bar{1}}\acute{\xi}^{1} + \acute{\chi}^{\bar{1}}\acute{\chi}^{1})_{(\pm)} \\
(\acute{\rho}\acute{u}_{\bar{1}2})_{(\pm)} &= -(\acute{\xi}^{\bar{2}}\acute{\xi}^{1} + \acute{\chi}^{\bar{2}}\acute{\chi}^{1})_{(\pm)} \\
(\acute{\rho}\acute{u}_{\bar{2}1})_{(\pm)} &= -(\acute{\xi}^{\bar{1}}\acute{\xi}^{2} + \acute{\chi}^{\bar{1}}\acute{\chi}^{2})_{(\pm)}
\end{aligned}
\qquad (8\text{-}32g)
$$

where the vertical bars in (8-32e) indicate the modulus of the invariant $(\acute{\xi}_{\alpha}\acute{\chi}^{\alpha})_{(\pm)}$ which, if (8-1d) is not satisfied, will be complex (although the imaginary part will be extremely small for macroscopic problems).

Note that we have written the electromagnetic field equation (8-32c) in terms of the irreducible form $A_{\bar{\alpha}\beta}$ of the 4-potential instead of the usual form A^{j} because it is $A_{\bar{\alpha}\beta}$ that appears in the spinor equations (8-31).

In (8-32a) and (8-32b) both the differential and integrated forms of the "thermal field equation" have been given. For problems involving harmonic expansions, the differential form (8-32a) would probably be more convenient, but for the purposes of the present discussion we shall refer to the integrated form (8-32b).

It will be noted from (8-32) that the source terms of all the field equations are algebraically nonlinear expressions of the spinor components; but, given the functional dependence of the spinor components on the space-time coordinates, it is a simple matter to determine the source terms and then solve the field equations. (Working with (8-32b), one would use the binomial theorem to generate a series involving the ratio of the higher harmonics in the harmonic expansion for ρ to the lowest harmonic, and then terminate the series after enough terms to provide sufficient accuracy.) The fields found by this procedure could then be substituted into the spinor equations (8-31) in order to calculate spinor functions more accurate than those used to calculate the fields. The solution of the equations (8-31) for given $\acute{g}$, $\acute{h}$, and $\acute{A}_{\bar{\alpha}\beta}$ is no problem since the equations are linear. This procedure is the basis for a straight-forward iteration solution of the complete magnetogas-dynamical problem. The equations (8-31) have been cast into a form in which all the interaction terms are contained in the square brackets preceded by the coupling constant $\alpha = \frac{1}{137}$. The smallness of this constant encourages the hope that the convergence of the solution would be rapid.

It has been noted that the gauge function $\acute{n}_{(\pm)}$ is completely arbitrary, and its presence in the spinor equations is a great asset since it can be chosen partially to cancel out unwanted higher harmonics in $\acute{g}$ and $\acute{h}_{(\pm)}$ which necessarily arise because the source terms of the field equations are algebraically nonlinear functions of the spinor components. To a lesser degree the constants $\acute{H}_{(+)}$ and $\acute{H}_{(-)}$ may also be adjusted to minimize the effects of higher harmonics, but we must recall that the choice of these two constants is restricted by the constraint (6-9) which in dimensionless form reads

$$\acute{m}_{(-)} \acute{H}_{(-)} \approx \acute{m}_{(+)} \acute{H}_{(+)} \tag{8-33}$$

How close this relation must come to being an exact equality depends on the magnitude of the difference between $\rho_o^{(-)}$ and $\rho_o^{(+)}$, and how great a difference between the temperatures of the electron and proton gases one is willing to regard as physically admissible.

It has already been noted that there is no need that $\acute{\eta}_{(+)}$ and $\acute{\eta}_{(-)}$ be equal. Moreover, we are free to choose different $\acute{\eta}_{(\pm)}$ and $\acute{H}_{(\pm)}$ at each different stage of the iteration. It may even be possible in certain time-independent problems to choose $\acute{\eta}_{(\pm)}$ and $\acute{H}_{(\pm)}$ in such a way as to allow a separation of the angular and radial dependence of the equations. In such a case an iteration procedure would be unnecessary since the system of partial differential equations would reduce to a nonlinear system of ordinary differential equations which could be solved by numerical means.

In order to make the significance of $\acute{\eta}_{(\pm)}$ clearer, it has been shown in Appendix C that introducing a gauge transformation characterized by the real scalar function $\acute{\eta}_{(\pm)}$ is equivalent to multiplying $\acute{\xi}^{\beta}_{(\pm)}$ by $e^{-i\alpha\acute{\eta}_{(\pm)}}$ and $\acute{\chi}^{\beta}_{(\pm)}$ by $e^{i\alpha\acute{\eta}_{(\pm)}}$.

APPENDIX A: ENERGY INJECTION AND VISCOSITY

Energy Injection

In Section II we saw that the possibility existed of taking energy injection into account by choosing an appropriate functional form for $Q^j_{(\pm)}$. Let us postulate the following form for $Q^j_{(\pm)}$:

$$Q^j_{(\pm)} = (\Lambda c u^j)_{(\pm)} \tag{A-1}$$

where $\Lambda_{(\pm)}$ is a scalar function of the fluid coordinates whose physical significance is discussed below. Substituting (A-1) into (2-22), we find that Euler's equation in the fluid rest-frame has the following form:

$$\left[\rho \frac{\partial(\breve{m}c^2)}{\partial\tau}\right]_{(\pm)} = \left[\rho m \frac{\partial g}{\partial\tau} + \frac{\partial P}{\partial\tau} + \rho(\Lambda m c^2)\right]_{(\pm)} \tag{A-2a}$$

$$\left\{\rho\left[\frac{\partial(\breve{m}\mathring{\vec{v}})}{\partial\tau}\right]_{v=0}\right\}_{(\pm)} = -\left[\rho m \mathring{\vec{\nabla}} g + \mathring{\vec{\nabla}} P\right]_{(\pm)} \pm q\,\rho_{(\pm)}\mathring{\vec{E}} \tag{A-2b}$$

Thus the choice (A-1) produces no new 3-force in the fluid rest-frame, but it does produce a new term $\rho(\Lambda mc^2)$ in (A-2a). This corresponds to an energy injection into the fluid in which Λ represents the fraction of the particle rest-energy that is injected per particle per unit time. Obviously Λ has the units of inverse time.

In general, $(\Lambda mc^2)_{(+)} \neq (\Lambda mc^2)_{(-)}$. The reason for this is that the nuclear reactions give the injected energy to the protons rather than to the electrons, and it is only through collisions that the electrons receive their share of the energy. If, however, we assume that energy equipartition establishes itself instantaneously (i.e., in a time very short compared with the time for the fluid to move a significant distance), then

$$(\Lambda m)_{(+)} = (\Lambda m)_{(-)} \qquad \text{(instantaneous equipartition)} \qquad \text{(A-3)}$$

Although energy injection introduces no new 3-force in the fluid rest-frame, we note from (2-22b) and (A-1) that in the observer's reference frame a 3-force $(\rho u^0)(\Lambda m\vec{v})$ appears, where $\Lambda m\vec{v}$ is just the momentum per particle that is injected along with the energy. (If the particle energy, hence its mass, were increased without the supplementary force $\Lambda m\vec{v}$, then, by Newton's Second Law, the particle would necessarily slow down.)

Obviously, (A-1) applies to the case of energy loss (through radiation), as well as to energy injection. It is only necessary to make Λ negative instead of positive. For negative Λ, the force $\Lambda m\vec{v}$ represents a drag on the particle. This is the radiation drag that occurs when a charged particle loses energy by radiation.

As was emphasized in Section II, the use of (A-1) instead of the adiabatic condition (2-30) makes no difference in the form of Euler's equation written in the form (2-24). It does, however, change the "thermal field equation". From (2-15) and (A-1) we see that, instead of (2-31), we have

$$\left[\rho \partial^j h - (\gamma - 1) h \partial^j \rho - \gamma \Lambda c (\rho u^j_{;j})\right]_{(\pm)} = 0 \tag{A-4}$$

Viscosity

For the sake of illustration, a rough and simple way will be given to take into account the viscous interaction between the two charged fluids. Let

$$Q^j_{(\pm)} = \frac{F}{(\rho m)_{(\pm)}} \left[(\rho u^j)_{(\mp)} - (\rho u^j)_{(\pm)}\right] \tag{A-5}$$

where F is a suitably chosen invariant function of the fluid variables which has the dimensions of a force.

Now we note that for the proton gas, for example, the term $(\rho m c Q^0)_{(+)}$ on the right side of (2-22a), which represents the rate of energy injection per unit volume, has the following form:

$$(\rho m c Q^0)_{(+)} = cF\left[(\rho u^0)_{(-)} - (\rho u^0)_{(+)}\right] \tag{A-6}$$

The quantity in the brackets is the total change density which, because of the electrostatic screening effect, must be everywhere very close to zero. Thus, the energy injection is also very close to zero. The energy injection for the electron gas is just the negative of (A-6). Thus the energy gained by one field is lost by the other, and the total energy generated is zero.

We note that the force term $(\rho m \vec{\Phi})_{(+)}$ on the right side of (2-22b), which represents the viscous force on the proton gas, has the form

$$(\rho m \vec{\Phi})_{(+)} = \frac{F}{c}\left[(\rho u^\circ \vec{v})_{(-)} - (\rho u^\circ \vec{v})_{(+)}\right] \tag{A-7}$$

The second term in the brackets represents a dragging force that acts to slow the proton gas down. The first term is a force which tends to accelerate the proton gas in the direction in which the electron gas is moving. These are just the effects we would expect viscosity to produce. The viscous force acting on the electron gas is just the negative of the viscous force acting on the proton gas. If, for any reason, we wished to free ourselves of this constraint, we could do so very easily by modifying (A-5) so that $\Phi^j_{(+)}$ and $\Phi^j_{(-)}$ involved different invariant functions $F_{(+)}$ and $F_{(-)}$. In this case, the total energy injected into both fields at each point in space by means of the viscous forces is not in general zero.

From (2-15), we see that when $\varphi^j_{(\pm)}$ is given by (A-5) the "thermal field equation" has the following form:

$$\left[\rho\,\partial^j h - (\gamma-1)\,h\,\partial^j \rho + \frac{\gamma F}{m}(\rho u^j)\right]_{(\pm)} = F\,\gamma_{(\pm)}\left[\frac{(\rho u^j)}{m}\right]_{(\mp)} \qquad \text{(A-8)}$$

The approximation based on (A-5) does not take into account the self-interaction of either fluid with itself that results because of viscosity. To do this we would have to introduce derivatives of the velocities into (A-5). In particular, we would introduce a term involving a suitable covariant generalization of the expression $(\vec{\nabla}\cdot\vec{\nabla})\vec{w}_{(\pm)}$.

APPENDIX B: EXPLICIT EXPRESSIONS FOR THE PARTICLE SPINORS

We saw in Section VII that the particle tetrapod has seven degrees of freedom, one of which, the particle density ρ, drops out if we work with a tetrapod normalized to unity. The remaining six degrees of freedom are to be identified with the three components of the particle velocity, and the three Euler angles needed to specify the orientation of the particle tripod in the rest-frame of the particle.

Rather than use the three Cartesian components of the particle velocity, we shall use $\beta = v/c$ and the two spherical angles $\hat{\theta}$ and $\hat{\varphi}$ that specify the direction of the particle velocity in the observer's reference frame. (The caret over $\hat{\theta}$ and $\hat{\varphi}$ merely indicates that these angles refer to the velocity, and distinguishes them from the angles $\tilde{\theta}, \tilde{\varphi}$ which will be used to specify the orientation of the particle tripod.) Thus we write the particle 4-velocity as follows:

$$
\begin{aligned}
u^0 &= (1-\beta^2)^{-1/2} \; ; \quad \beta = v/c \\
u^1 &= u^0 \beta \sin\hat{\theta} \cos\hat{\varphi} \\
u^2 &= u^0 \beta \sin\hat{\theta} \sin\hat{\varphi} \\
u^3 &= u^0 \beta \cos\hat{\theta}
\end{aligned}
\qquad \text{(B-1)}
$$

(There will be no need in this appendix for (+) and (-) subscripts since it is understood that everything applies equally well to either of the two charged fluids.)

In order to specify the angular orientation of the particle tripod, we imagine that the observer accelerates himself and his coordinate system in the direction of the particle velocity until his velocity is the same as that of the particle, i.e., until he is in the particle rest-frame. This is accomplished without rotating his coordinate system. The orientation of the tripod $\mathring{\vec{k}}$, $\mathring{\vec{\lambda}}$, and $\mathring{\vec{\mu}}$ is then easily specified in terms of the Euler angles $\tilde{\varphi}$, $\tilde{\theta}$, and $\tilde{\psi}$ that would carry the observer's own x, y, and z axes over into the positions occupied by the $\mathring{\vec{k}}$, $\mathring{\vec{\lambda}}$, and $\mathring{\vec{\mu}}$ vectors respectively. Thus, we first rotate the observer's tripod of coordinates axes an angle $\tilde{\varphi}$ about his z axis, then an angle $\tilde{\theta}$ about his <u>y axis</u>, and finally an angle $\tilde{\psi}$ about the direction in which the observer's z axis finds itself after the first two rotations have been carried out. The observer's tripod, like the tripod $(\mathring{\vec{k}}, \mathring{\vec{\lambda}}, \mathring{\vec{\mu}})$ is assumed to constitute a right-handed system, and the positive directions for the rotations are given by the right-hand screw rule. Note that $\tilde{\theta}$ specifies the rotation about the <u>y axis</u>, rather than the <u>x axis</u>, as is frequently the case. The reason for this choice is that is makes the angular dependence of $\vec{\overset{\circ}{\mu}}$ similar to that of (u^1, u^2, u^3) in (B-1). Thus we have

$$\begin{aligned}
\overset{\circ}{\mu}{}^1 &= \sin\tilde{\theta}\cos\tilde{\varphi} \\
\overset{\circ}{\mu}{}^2 &= \sin\tilde{\theta}\sin\tilde{\varphi} \\
\overset{\circ}{\mu}{}^3 &= \cos\tilde{\theta}
\end{aligned} \tag{B-2a}$$

$$\begin{aligned}
\overset{\circ}{k}{}^1 &= \cos\tilde{\theta}\cos\tilde{\varphi}\cos\tilde{\psi} - \sin\tilde{\varphi}\sin\tilde{\psi} \\
\overset{\circ}{k}{}^2 &= \cos\tilde{\theta}\sin\tilde{\varphi}\cos\tilde{\psi} + \cos\tilde{\varphi}\sin\tilde{\psi} \\
\overset{\circ}{k}{}^3 &= -\sin\tilde{\theta}\cos\tilde{\psi}
\end{aligned} \tag{B-2b}$$

$$\begin{aligned}
\overset{\circ}{\lambda}{}^1 &= -(\cos\tilde{\theta}\cos\tilde{\varphi}\sin\tilde{\psi} + \sin\tilde{\varphi}\cos\tilde{\psi}) \\
\overset{\circ}{\lambda}{}^2 &= -(\cos\tilde{\theta}\sin\tilde{\varphi}\sin\tilde{\psi} - \cos\tilde{\varphi}\cos\tilde{\psi}) \\
\overset{\circ}{\lambda}{}^3 &= \sin\tilde{\theta}\sin\tilde{\psi}
\end{aligned} \tag{B-2c}$$

The expressions (B-2) are valid in the particle rest-frame. It is now a straight-forward matter to transform these three vectors to the observer's original frame of reference. Let $\vec{v}_1$ be the velocity unit 3-vector, i.e.

$$\vec{v}_1 = \vec{v}/v = (\sin\hat{\theta}\cos\hat{\varphi},\ \sin\hat{\theta}\sin\hat{\varphi},\ \cos\hat{\theta}) \tag{B-3}$$

Then from the Lorentz transformation we have

$$\mu^\circ = u^\circ \beta\,(\vec{\mathring{\mu}} \cdot \vec{n}_1) \tag{B-4a}$$

$$\vec{\mu} = (\mu^1, \mu^2, \mu^3) = \vec{\mathring{\mu}} + (u^\circ - 1)(\vec{\mathring{\mu}} \cdot \vec{n}_1)\,\vec{n}_1 \tag{B-4b}$$

where

$$\vec{\mathring{\mu}} \cdot \vec{n}_1 = \sin\hat{\theta}\,\sin\tilde{\theta}\,\cos(\hat{\varphi} - \tilde{\varphi}) + \cos\hat{\theta}\,\cos\tilde{\theta} \tag{B-4c}$$

$$k^\circ = u^\circ \beta\,(\vec{\mathring{k}} \cdot \vec{n}_1) \tag{B-5a}$$

$$\vec{k} = (k^1, k^2, k^3) = \vec{\mathring{k}} + (u^\circ - 1)(\vec{\mathring{k}} \cdot \vec{n}_1)\,\vec{n}_1 \tag{B-5b}$$

where

$$\vec{\mathring{k}} \cdot \vec{n}_1 = \sin\hat{\theta}\,\sin(\hat{\varphi} - \tilde{\varphi})\,\sin\tilde{\psi} + \sin\hat{\theta}\,\cos\tilde{\theta}\,\cos(\hat{\varphi} - \tilde{\varphi})\,\cos\tilde{\psi} - \cos\hat{\theta}\,\sin\tilde{\theta}\,\cos\tilde{\psi} \tag{B-5c}$$

$$\lambda^\circ = u^\circ \beta\,(\vec{\mathring{\lambda}} \cdot \vec{n}_1) \tag{B-6a}$$

$$\vec{\lambda} = (\lambda^1, \lambda^2, \lambda^3) = \vec{\mathring{\lambda}} + (u^\circ - 1)(\vec{\mathring{\lambda}} \cdot \vec{n}_1)\,\vec{n}_1 \tag{B-6b}$$

where

$$\vec{\mathring{\lambda}} \cdot \vec{n}_1 = \sin\hat{\theta}\,\sin(\hat{\varphi} - \tilde{\varphi})\cos\tilde{\psi} - \sin\hat{\theta}\,\cos(\hat{\varphi} - \tilde{\varphi})\cos\tilde{\theta}\,\sin\tilde{\psi} + \cos\hat{\theta}\,\sin\tilde{\theta}\,\sin\tilde{\psi} \tag{B-6c}$$

By means of (B-1) - (B-6), all sixteen components of the tetrapod (normalized to unity) have been specified as functions of β, $\hat{\theta}$, $\hat{\varphi}$, $\tilde{\theta}$, $\tilde{\varphi}$, and $\tilde{\psi}$. We must now find expressions for the components of ξ^α and χ^α in terms of these variables (plus ρ and ϵ) such that the relations (7-27) are satisfied. It can be verified by direct calculation that the following are the desired expressions:

$$\xi^\alpha = 2^{-1/4} \rho^{1/2} e^{\frac{i}{2}(\epsilon - \tilde{\psi})} \hat{\xi}^\alpha \tag{B-7a}$$

$$\chi^\alpha = 2^{-1/4} \rho^{1/2} e^{\frac{i}{2}(\epsilon + \tilde{\psi})} \hat{\chi}^\alpha \tag{B-7b}$$

where

$$\hat{\xi}^1 = \left[\cosh \nu/2 + \sinh \nu/2 \cos\hat{\theta}\right] \cos \tilde{\theta}/2 \, e^{-i\tilde{\varphi}/2} + \sinh \nu/2 \sin\hat{\theta} \sin\tilde{\theta}/2 \, e^{-i(\hat{\varphi} - \tilde{\varphi}/2)} \tag{B-8a}$$

$$\hat{\xi}^2 = \left[\cosh \nu/2 - \sinh \nu/2 \cos\hat{\theta}\right] \sin \tilde{\theta}/2 \, e^{i\tilde{\varphi}/2} + \sinh \nu/2 \sin\hat{\theta} \cos\tilde{\theta}/2 \, e^{i(\hat{\varphi} - \tilde{\varphi}/2)} \tag{B-8b}$$

$$\hat{\chi}^1 = -\left[\cosh \nu/2 + \sinh \nu/2 \cos\hat{\theta}\right] \sin \tilde{\theta}/2 \, e^{-i\tilde{\varphi}/2} + \sinh \nu/2 \sin\hat{\theta} \sin\tilde{\theta}/2 \, e^{-i(\hat{\varphi} - \tilde{\varphi}/2)} \tag{B-8c}$$

$$\hat{\chi}^2 = \left[\cosh \nu/2 - \sinh \nu/2 \cos\hat{\theta}\right] \cos \tilde{\theta}/2 \, e^{i\tilde{\varphi}/2} - \sinh \nu/2 \sin\hat{\theta} \sin\tilde{\theta}/2 \, e^{i(\hat{\varphi} - \tilde{\varphi}/2)} \tag{B-8d}$$

where

$$\nu = \tanh^{-1} \beta \tag{B-8e}$$

and

$$\begin{aligned} \cosh \nu/2 &= \sqrt{\tfrac{1}{2}(u^{0}+1)} \\ \sinh \nu/2 &= \sqrt{\tfrac{1}{2}(u^{0}-1)} \end{aligned} \tag{B-8f}$$

Let us now find expressions for the variables $\rho, \epsilon, \beta, \hat{\theta}$, $\hat{\varphi}, \tilde{\theta}, \tilde{\varphi}$, and $\tilde{\psi}$ in terms of the spinors ξ^{α} and χ^{α}. From (7-35) we have

$$\rho = \left[2(\xi_{\alpha}\chi^{\alpha})(\xi_{\bar{\beta}}\chi^{\bar{\beta}})\right]^{1/2} \tag{B-9}$$

and

$$e^{i\epsilon} = \left[\frac{\xi_{\alpha}\chi^{\alpha}}{\xi_{\bar{\beta}}\chi^{\bar{\beta}}}\right]^{1/2} \tag{B-10}$$

In order to find expressions for the other variables, it is convenient to introduce the functions A and B defined as follows:

$$A = \frac{\xi^{1}}{\sqrt{\xi_{\alpha}\chi^{\alpha}}} + \frac{\chi^{\bar{2}}}{\sqrt{\xi_{\bar{\beta}}\chi^{\bar{\beta}}}} = \sqrt{2(u^{0}+1)}\, \cos \hat{\theta}/2\, e^{-\frac{i}{2}(\tilde{\psi}+\hat{\varphi})} \tag{B-11a}$$

$$B = \frac{\xi^{2}}{\sqrt{\xi_{\alpha}\chi^{\alpha}}} - \frac{\chi^{\bar{1}}}{\sqrt{\xi_{\bar{\beta}}\chi^{\bar{\beta}}}} = \sqrt{2(u^{0}+1)}\, \sin \tilde{\theta}/2\, e^{-\frac{i}{2}(\tilde{\psi}-\hat{\varphi})} \tag{B-11b}$$

From (B-11) it follows that

$$\cos \tilde{\Theta} = \frac{A\bar{A} - B\bar{B}}{A\bar{A} + B\bar{B}} \tag{B-12}$$

$$e^{i\tilde{\varphi}} = \left[\frac{\bar{A}\,B}{A\,\bar{B}}\right]^{1/2} \tag{B-13}$$

$$e^{i\tilde{\psi}} = \left[\frac{\bar{A}\,\bar{B}}{A\,B}\right]^{1/2} \tag{B-14}$$

In order to find β, $\hat{\Theta}$, and $\hat{\varphi}$, it is most convenient to use the following relations, which result from (7-26), (7-25), and (7-22):

$$\rho u^0 = \frac{1}{\sqrt{2}}\left[\rho^1\rho^{\bar{1}} + \rho^2\rho^{\bar{2}} + \chi^1\chi^{\bar{1}} + \chi^2\chi^{\bar{2}}\right] \tag{B-15a}$$

$$\rho u^1 = \rho u^0\beta \sin\hat{\Theta}\cos\hat{\varphi} = \sqrt{2}\,\mathcal{R}\left[\rho^1\rho^{\bar{2}} + \chi^1\chi^{\bar{2}}\right] \tag{B-15b}$$

$$\rho u^2 = \rho u^0\beta \sin\hat{\Theta}\sin\hat{\varphi} = -\sqrt{2}\,\mathcal{I}\left[\rho^1\rho^{\bar{2}} + \chi^1\chi^{\bar{2}}\right] \tag{B-15c}$$

$$\rho u^3 = \rho u^0\beta \cos\hat{\Theta} = \frac{1}{\sqrt{2}}\left[\rho^1\rho^{\bar{1}} - \rho^2\rho^{\bar{2}} + \chi^1\chi^{\bar{1}} - \chi^2\chi^{\bar{2}}\right] \tag{B-15d}$$

Having found u^0 from (B-15a) and (B-9), we use the following relation to find :

$$\beta = \frac{1}{u^0}\left[(u^0)^2 - 1\right]^{1/2} \tag{B-17}$$

From (B-15b) - (B-15d) we have

$$\tan\hat{\Theta} = \frac{1}{u^3}\left[(u^1)^2 + (u^2)^2\right]^{1/2} \tag{B-18}$$

$$\tan\hat{\varphi} = u^2/u^1 \tag{B-19}$$

In the solution of a magnetogas-dynamical problem of the self-excited dynamo type, we would solve the spinor equations (8-1) or (8-31) and arrive at expressions for ξ^{α} and χ^{α} as functions of time t and the spherical coordinates r, θ, and φ. We would then use the above expressions to find ρ, β, $\hat{\theta}$, $\hat{\varphi}$, $\tilde{\theta}$, $\tilde{\varphi}$, and $\tilde{\psi}$ as functions of the coordinates (t, r, θ, φ).

APPENDIX C: DIFFERENTIAL SPINOR RELATIONS

Observable Quantities

Using the spinors ξ^{α} and χ^{α}, and the gradient operator $\partial_{\bar{\alpha}\beta}$ (or ∂_j) as the basic ingredients, it is possible to construct many differential expressions. Before any physical meaning can be given to these expressions, however, it is necessary to translate them into expressions involving the observable quantities ρ, ϵ, u^j, μ^j, ω^j, and s^{jk} which are defined as follows:

$$\rho e^{i\epsilon} \equiv \sqrt{2}\,(\xi_{\alpha}\chi^{\alpha}) \tag{C-1}$$

$$\rho u^j \equiv \sigma^j_{\bar{\alpha}\beta}(\xi^{\beta}\xi^{\bar{\alpha}} + \chi^{\beta}\chi^{\bar{\alpha}}) \tag{C-2}$$

$$\rho \mu^j \equiv \sigma^j_{\bar{\alpha}\beta}(\xi^{\beta}\xi^{\bar{\alpha}} - \chi^{\beta}\chi^{\bar{\alpha}}) \tag{C-3}$$

$$\omega^j \equiv \tfrac{1}{2}\lambda_{\ell}\,\partial^j k^{\ell} \tag{C-4a}$$

where

$$\rho k^j \equiv \sigma^j_{\bar{\alpha}\beta}(\xi^{\beta}\chi^{\bar{\alpha}} + \chi^{\beta}\xi^{\bar{\alpha}}) \tag{C-4b}$$

$$\rho \lambda^j \equiv -i\sigma^j_{\bar{\alpha}\beta}(\xi^{\beta}\chi^{\bar{\alpha}} - \chi^{\beta}\xi^{\bar{\alpha}}) \tag{C-4c}$$

Using (C-4b) and (C-4c) in (C-4a), ω^j can also be written as follows:

$$\omega^j = \mathcal{I}\left[\left(\frac{\chi^\beta}{\sqrt{\xi_\alpha\chi^\alpha}}\right)\partial^j\left(\frac{\xi_\beta}{\sqrt{\xi_\alpha\chi^\alpha}}\right)\right] \tag{C-5a}$$

or

$$\omega^j = \tfrac{1}{2}\mathcal{I}\left[(\xi_\alpha\chi^\alpha)^{-1}(\chi^\beta\partial^j\xi_\beta - \xi_\beta\partial^j\chi^\beta)\right] \tag{C-5b}$$

The dual tensors $\mathcal{A}^{jk}$ and $\hat{\mathcal{A}}^{jk}$ are defined as follows:

$$\mathcal{A}^{jk} \equiv -\varepsilon^{jk\ell n}u_\ell \mu_n = -\tfrac{1}{2}\varepsilon^{jk\ell n}\hat{\mathcal{A}}_{\ell n} \tag{C-6}$$

$$\hat{\mathcal{A}}^{jk} \equiv \tfrac{1}{2}\varepsilon^{jk\ell n}\mathcal{A}_{\ell n} = u^j\mu^k - u^k\mu^j \tag{C-7}$$

The following orthogonality properties result from (C-6):

$$\mathcal{A}^{jk}u_k = u_j\mathcal{A}^{jk} = 0 \tag{C-8a}$$

$$\mathcal{A}^{jk}\mu_k = \mu_j\mathcal{A}^{jk} = 0 \tag{C-8b}$$

It can be shown that $\mathcal{A}^{jk}$ and $\hat{\mathcal{A}}^{jk}$ are related to the spinors ξ^α and χ^α as follows:

$$\hat{\mathcal{A}}^{jk} = \mathcal{R}\left[(\xi_\nu\chi^\nu)^{-1}g^{k\ell}\sigma_\ell^{\beta\bar\lambda}\sigma^j_{\bar\lambda\alpha}(\xi^\alpha\chi_\beta + \chi^\alpha\xi_\beta)\right] \tag{C-9}$$

$$\mathcal{A}^{jk} = -\mathcal{I}\left[(\xi_\nu\chi^\nu)^{-1}g^{k\ell}\sigma_\ell^{\beta\bar\lambda}\sigma^j_{\bar\lambda\alpha}(\xi^\alpha\chi_\beta + \chi^\alpha\xi_\beta)\right] \tag{C-10}$$

Thus we see that the symmetric spinor expression $(\xi^\alpha\chi_\beta + \chi^\alpha\xi_\beta)$ is to be assocated with the dual tensors $\hat{\mathcal{A}}^{jk}$ and $\mathcal{A}^{jk}$. The antisymmetric spinor expression $(\xi^{\alpha}\chi_{\beta} - \chi^{\alpha}\xi_{\beta})$ corresponds to the Kronecker delta function δ_j^k as shown by the following identity:

$$\delta_j^k = -(\xi_{\nu}\chi^{\nu})^{-1}\,\sigma_j{}^{\beta\bar{\lambda}}\sigma^k_{\bar{\lambda}\alpha}\,(\xi^{\alpha}\chi_{\beta} - \chi^{\alpha}\xi_{\beta}) \tag{C-11}$$

In this appendix we shall have no need to label quantities with the (+) and (-) subscripts. When the sign of the particle charge enters an equation it will be indicated by σ, where $\sigma = \pm 1$.

Gradients of Spinors

Using the definitions (C-1) to (C-5) it can be verified that the following expressions for the gradients of ξ^{β} and χ^{β} are valid identities:

$$\begin{aligned}\partial_j \xi^{\beta} = \xi^{\beta}\left[\tfrac{1}{2}(\partial_j \ln\rho + \mu_{\ell}\partial_j u^{\ell}) + i(\omega_j + \tfrac{1}{2}\partial_j\epsilon)\right] \\ - \frac{e^{i\epsilon}}{\sqrt{2}}\chi_{\bar{\alpha}}\,\partial_j(u^{\beta\bar{\alpha}} + \mu^{\beta\bar{\alpha}})\end{aligned} \tag{C-12a}$$

$$\begin{aligned}\partial_j \chi^{\beta} = \chi^{\beta}\left[\tfrac{1}{2}(\partial_j \ln\rho - \mu_{\ell}\partial_j u^{\ell}) - i(\omega_j - \tfrac{1}{2}\partial_j\epsilon)\right] \\ + \frac{e^{i\epsilon}}{\sqrt{2}}\xi_{\bar{\alpha}}\,\partial_j(u^{\beta\bar{\alpha}} - \mu^{\beta\bar{\alpha}})\end{aligned} \tag{C-12b}$$

These identities can be written in a more symmetric form, as follows:

$$\partial_k \xi^{\beta} = -\sqrt{2}\,e^{i\epsilon}\sigma_j{}^{\beta\bar{\lambda}}\chi_{\bar{\lambda}}\left[(2\rho)^{-1}\partial_k(\rho\mu^j) + i(u^j\omega_k + \tfrac{1}{2}\mu^j\partial_k\epsilon - \tfrac{1}{2}s^{j\ell}\partial_k u_{\ell})\right] \tag{C-13a}$$

$$\partial_k \chi^\beta = -\sqrt{2}\, e^{i\epsilon} \sigma_j^{\ \beta\bar{\lambda}} \xi_{\bar{\lambda}} \Big[(2\rho)^{-1} \partial_k(\rho\mu^j) + i(u^j\omega_k + \tfrac{1}{2}\mu^j \partial_k \epsilon - \tfrac{1}{2} s^{j\ell} \partial_k u_\ell)\Big] \qquad \text{(C-13b)}$$

If we multiply these two identities by $\sigma^k_{\bar{\alpha}\beta}$, contracting the k and the β, we arrive at the following identities:

$$\partial_{\bar{\alpha}\beta} \xi^\beta = \xi^\beta \sigma^k_{\bar{\alpha}\beta} \Big[\tfrac{1}{2}(\partial_k \ln\rho - \mu_j \partial_k u^j - \partial^j \hat{s}_{jk} + s_{kj}\partial^j \epsilon) + i(\omega_k + \tfrac{1}{2}\partial^j s_{jk} + \tfrac{1}{2}\hat{s}_{kj}\partial^j\epsilon)\Big] \qquad \text{(C-14a)}$$

$$\partial_{\bar{\alpha}\beta} \chi^\beta = \chi^\beta \sigma^k_{\bar{\alpha}\beta} \Big[\tfrac{1}{2}(\partial_k \ln\rho + \mu_j \partial_k u^j + \partial^j \hat{s}_{jk} - s_{kj}\partial^j \epsilon) - i(\omega_k + \tfrac{1}{2}\partial^j s_{jk} + \tfrac{1}{2}\hat{s}_{kj}\partial^j\epsilon)\Big] \qquad \text{(C-14b)}$$

Alternative expressions for $\partial_{\bar{\alpha}\beta}\xi^\beta$ and $\partial_{\bar{\alpha}\beta}\chi^\beta$ could be found by contracting (C-12) instead of (C-13).

Using (C-12) - (C-14), it is possible to derive the following identities:

Scalar Identities

$$\xi^{\bar{\alpha}} \partial_{\bar{\alpha}\beta} \xi^\beta - \chi^{\bar{\alpha}} \partial_{\bar{\alpha}\beta} \chi^\beta = \tfrac{1}{2} \partial_j(\rho\mu^j) + i\, u_j \Big[\rho\omega^j + \tfrac{1}{2}\partial_k(\rho s^{kj}) + \tfrac{1}{2}\rho \hat{s}^{jk} \partial_k \epsilon\Big] \qquad \text{(C-15)}$$

$$\xi^{\bar{\alpha}} \partial_{\bar{\alpha}\beta} \xi^\beta + \chi^{\bar{\alpha}} \partial_{\bar{\alpha}\beta} \chi^\beta = \tfrac{1}{2} \partial_j(\rho u^j) + i\mu_j \Big[\rho\omega^j + \tfrac{1}{2}\partial_k(\rho s^{kj}) + \tfrac{1}{2}\rho \hat{s}^{jk} \partial_k \epsilon\Big] \qquad \text{(C-16)}$$

Vector Identities

$$
\begin{aligned}
&g^{j\ell}\sigma_\ell{}^{\nu\bar\alpha}(\chi_\nu \partial_{\bar\alpha\beta}\xi^\beta - \xi_\nu \partial_{\bar\alpha\beta}\chi^\beta) = \\
&= -\frac{e^{i\epsilon}}{\sqrt{2}}\left\{s^{jk}\left[\rho\omega_k + \tfrac{1}{2}\partial^\ell(\rho s_{\ell k})\right] + \tfrac{1}{2}u^j\partial_k(\rho u^k) - \tfrac{1}{2}\mu^j\partial_k(\rho\mu^k)\right\} \\
&\quad -\frac{i e^{i\epsilon}}{2\sqrt{2}}\left\{\rho\partial^j\epsilon + 2\hat{s}^{jk}\left[\rho\omega_k + \tfrac{1}{2}\partial^\ell(\rho s_{\ell k})\right]\right. \\
&\qquad\qquad \left. + \rho s^{jk}\left[u^\ell\partial_\ell\mu_k - \mu^\ell(\partial_\ell u_k - \partial_k u_\ell)\right]\right\}
\end{aligned}
\tag{C-17}
$$

$$
\begin{aligned}
&g^{j\ell}\sigma_\ell{}^{\nu\bar\alpha}(\chi_\nu \partial_{\bar\alpha\beta}\xi^\beta + \xi_\nu \partial_{\bar\alpha\beta}\chi^\beta) = \\
&= \frac{e^{i\epsilon}}{2\sqrt{2}}\left\{\rho u^k\partial_k\mu^j + \rho\mu_k\left[(\partial^j u^k - \partial^k u^j) - \varepsilon^{jk\ell n}u_\ell\partial_n\epsilon\right]\right. \\
&\qquad\qquad \left. + \mu^j\partial_k(\rho u^k) - u^j\partial_k(\rho\mu^k)\right\} \\
&\quad -\frac{i e^{i\epsilon}}{\sqrt{2}}\left\{\rho\omega^j + \tfrac{1}{2}\partial_k(\rho s^{kj}) + \tfrac{1}{2}\rho\hat{s}^{jk}\partial_k\epsilon\right\}
\end{aligned}
\tag{C-18}
$$

Tensor Identities

$$
\begin{aligned}
&g^{j\ell}\sigma_\ell{}^{\beta\bar\nu}\sigma_{\bar\nu\alpha}{}^k(\xi^\alpha\partial_{\bar\lambda\beta}\xi^{\bar\lambda} - \chi^\alpha\partial_{\bar\lambda\beta}\chi^{\bar\lambda}) = \\
&= \tfrac{1}{2}g^{jk}\left\{\tfrac{1}{2}\partial_\ell(\rho\mu^\ell) - iu_n\left[\rho\omega^n + \tfrac{1}{2}\partial_\ell(\rho s^{\ell n}) + \tfrac{1}{2}\rho\hat{s}^{n\ell}\partial_\ell\epsilon\right]\right\} \\
&\quad + \tfrac{1}{2}\left\{\tfrac{1}{2}\left[\partial^j(\rho\mu^k) - \partial^k(\rho\mu^j)\right] - \varepsilon^{jk\ell n}\mathfrak{J}_{\ell n}\right\} \\
&\quad -\frac{i}{4}\left\{\begin{aligned}
&s^{jk}\partial_\ell(\rho u^\ell) + 2\left[\rho\omega^j + \tfrac{1}{2}\partial_\ell(\rho s^{\ell j}) + \tfrac{1}{2}\rho\hat{s}^{j\ell}\partial_\ell\epsilon\right]u^k \\
&\qquad\qquad - 2\left[\rho\omega^k + \tfrac{1}{2}\partial_\ell(\rho s^{\ell k}) + \tfrac{1}{2}\rho\hat{s}^{k\ell}\partial_\ell\epsilon\right]u^j \\
&+ \rho u^\ell\partial_\ell s^{jk} - \rho s^j{}_\ell\left[(\partial^\ell u^k - \partial^k u^\ell) - \varepsilon^{\ell kpq}u_p\partial_q\epsilon\right] \\
&\qquad\qquad + \rho s^k{}_\ell\left[(\partial^\ell u^j - \partial^j u^\ell) - \varepsilon^{\ell jpq}u_p\partial_q\epsilon\right]
\end{aligned}\right\}
\end{aligned}
\tag{C-19}
$$

$$
\begin{aligned}
&g^{j\ell}\sigma_\ell^{\beta\bar\nu}\sigma_{\bar\nu\alpha}^{k}\left(\xi^\alpha\partial_{\bar\lambda\beta}\xi^{\bar\lambda} + \chi^\alpha\partial_{\bar\lambda\beta}\chi^{\bar\lambda}\right) = \\
&= \tfrac12 g^{jk}\left\{\tfrac12\partial_\ell(\rho u^\ell) - i\mu_n\left[\rho\omega^n + \tfrac12\partial_\ell(\rho s^{\ell n}) + \tfrac12\rho\hat{s}^{n\ell}\partial_\ell\epsilon\right]\right\} \\
&\quad + \tfrac12\left\{\tfrac12\left[\partial^j(\rho u^k) - \partial^k(\rho u^j)\right] - \varepsilon^{jk\ell n}\mathcal{Q}_{\ell n}\right\} \\
&\quad - \frac{i}{4}\left\{\begin{aligned} & s^{jk}\partial_\ell(\rho\mu^\ell) + 2\left[\rho\omega^j + \tfrac12\partial_\ell(\rho s^{\ell j}) + \tfrac12\rho\hat{s}^{j\ell}\partial_\ell\epsilon\right]\mu^k \\ &\qquad - 2\left[\rho\omega^k + \tfrac12\partial_\ell(\rho s^{\ell k}) + \tfrac12\rho\hat{s}^{k\ell}\partial_\ell\epsilon\right]\mu^j \\ & + \rho\mu^\ell\partial_\ell s^{jk} - \rho s^j{}_\ell\left[(\partial^\ell\mu^k - \partial^k\mu^\ell) - \varepsilon^{\ell kpq}\mu_p\partial_q\epsilon\right] \\ &\qquad + \rho s^k{}_\ell\left[(\partial^\ell\mu^j - \partial^j\mu^\ell) - \varepsilon^{\ell jpq}\mu_p\partial_q\epsilon\right]\end{aligned}\right\}
\end{aligned}
\qquad \text{(C-20)}
$$

$$g_{j\ell}\,\sigma^{\ell}_{\bar\alpha\beta}\left(\xi^{\bar\alpha}\partial_k\xi^\beta - \chi^{\bar\alpha}\partial_k\chi^\beta\right) = \tfrac12\partial_k(\rho\mu_j) + i\,\mathcal{T}_{jk} \qquad \text{(C-21)}$$

$$g_{j\ell}\,\sigma^{\ell}_{\bar\alpha\beta}\left(\xi^{\bar\alpha}\partial_k\xi^\beta + \chi^{\bar\alpha}\partial_k\chi^\beta\right) = \tfrac12\partial_k(\rho u_j) + i\,\mathcal{Q}_{jk} \qquad \text{(C-22)}$$

where

$$\mathcal{T}_{jk} = \rho\left[u_j\omega_k + \tfrac12\mu_j\partial_k\epsilon - \tfrac12 s_{j\ell}\partial_k u^\ell\right] \qquad \text{(C-23)}$$

$$\mathcal{Q}_{jk} = \rho\left[\mu_j\omega_k + \tfrac12 u_j\partial_k\epsilon - \tfrac12 s_{j\ell}\partial_k\mu^\ell\right] \qquad \text{(C-24)}$$

Spinor Equations

The relations (C-15) - (C-24) are just identities based on the definitions (C-1) - (C-10). By introducing spinor equations of motion, and substituting these into the above identities, we can find the fluid equations of motion in terms of the observable quantities defined in (C-1) - (C-10). The spinor equations of motion are postulated to have the following form:

$$i\hbar\, \partial_{\bar{\alpha}\beta} \xi^{\beta} = \frac{\breve{m}c}{\sqrt{2}} \chi_{\bar{\alpha}} - \frac{q}{c} A_{\bar{\alpha}\beta} \xi^{\beta} \tag{C-25a}$$

$$i\hbar\, \partial_{\bar{\alpha}\beta} \chi^{\beta} = \frac{\breve{m}c}{\sqrt{2}} \xi_{\bar{\alpha}} + \frac{q}{c} A_{\bar{\alpha}\beta} \chi^{\beta} \tag{C-25b}$$

where

$$\breve{m} = m(1 + g/c^2 + h/c^2) \tag{C-25c}$$

We shall at times impose the following constraint on the solutions of (C-25):

$$\mathcal{I}(\xi_{\beta} \chi^{\beta}) = 0 \tag{C-26a}$$

which is equivalent to the constraint

$$\cos \epsilon = \sigma = \pm 1 \tag{C-26b}$$

or

$$\begin{aligned} \epsilon &= 0 \quad \text{for} \quad \sigma = +1 \\ \epsilon &= \pi \quad \text{for} \quad \sigma = -1 \end{aligned} \tag{C-26c}$$

where ϵ is the phase angle common to ξ^{β} and χ^{β} and σ is the sign of the particle charge. The symbol σ will always represent either

+1 or -1, but in much of what follows we shall not impose condition (C-26), with the result that in such cases $\cos\epsilon \neq \sigma$. When condition (C-26) is imposed, however, we note that the product $\sigma\cos\epsilon$, which occurs in some of the following expressions, equals +1.

Fluid Equations of Motion

We shall first derive the fluid equations of motion without imposing the constraint (C-26). In this way it will be possible to estimate the error introduced by neglecting this constraint.

Substituting (C-25) into (C-15) and (C-16), we have from the real parts of these identities the following relations:

$$\partial_j(\rho\mu^j) = \frac{2\rho}{\check{\lambda}}\sin\epsilon \tag{C-27a}$$

where

$$\check{\lambda} = \frac{\hbar}{\check{m}c} \tag{C-27b}$$

and

$$\partial_j(\rho u^j) = 0 \tag{C-28}$$

These equations hold for each of the two charged fluids individually. (C-28) is just the statement of charge conservation, or conservation of the number of particles, for each of the two fluids taken individually.

Equation (C-27) allows us to estimate the magnitude of ϵ for the case that we solve (C-25) without imposing the constraint (C-26). We note that, if we limit ourselves to solutions in which ρ and μ^j are smoothly varying functions whose values change significantly only over distances comparable with L and in time intervals greater than L/c, where L is a length that specifies the dimensions of the system, then from (C-27) we have

$$|\sin \epsilon| = \left| \frac{\check{\lambda}}{2\rho} \partial_j(\rho\mu^j) \right| \sim \frac{\check{\lambda}}{L} \approx \frac{\lambda}{L} \tag{C-29}$$

where in the last step we have replaced $\check{\lambda}$ by λ (defined in (7-47) in terms of the free, rather than the bound, particle mass) because the gravitational and thermal energy of a particle is small compared with its rest energy. From (C-29) we see that $|\sin \epsilon|$ is very small for macroscopic systems. Thus, for such systems, we have

$$\begin{aligned} |\epsilon| &\sim \frac{\lambda}{L} \quad \text{for} \quad \sigma = +1 \\ |\pi - \epsilon| &\sim \frac{\lambda}{L} \quad \text{for} \quad \sigma = -1 \end{aligned} \tag{C-30}$$

where the first case applies to positively charged particles, and the second case to negatively charged particles.

If we impose the condition (C-26), we have, instead of (C-27)

$$\partial_j(\rho\mu^j) = 0 \quad \text{for} \quad \cos \epsilon = \sigma \tag{C-31}$$

Substituting (C-25) into (C-18), we arrive at two real vector equations, one of which is the following:

$$\sigma\hbar\rho\omega^j = \rho(\sigma\cos\epsilon)\breve{m}cu^j - \frac{\sigma\hbar}{2}\partial_k(\rho s^{kj}) - \frac{\sigma\hbar}{2}\rho\hat{s}^{jk}\partial_k\epsilon + \frac{\rho(\sigma q)}{c}A^j \quad \text{(C-32)}$$

In order to gain an intuitive feeling for this equation, let us examine first the second term on the right side. Using (C-6), we can derive the following identity:

$$-\frac{\sigma\hbar}{2}\partial_k(\rho s^{kj}) = \rho\tilde{m}cu^j + \sigma\hbar\rho\pi^j \quad \text{(C-33a)}$$

where

$$\tilde{m} = -\frac{\sigma\hbar}{2c\rho}u_j\partial_k(\rho s^{kj}) \quad \text{(C-33b)}$$

and

$$\pi^j = (2\rho)^{-1}\varepsilon^{jkln}u_k\partial_l(\rho\mu_n) - \frac{1}{2}s^{jk}u^l\partial_l u_k \quad \text{(C-33c)}$$

We note that, because of (C-8a) and the antisymmetry of ε^{jkln}

$$\pi^j u_j = 0 \quad \text{(C-34)}$$

Let us evaluate $\sigma\hbar\rho\pi^j$ in the particle rest-frame:

$$\begin{aligned} \sigma\hbar\rho\mathring{\vec{\pi}} &= \left[\sigma\hbar\rho(\pi^1, \pi^2, \pi^3)\right]_{v=0} \\ &= \left[\vec{\nabla}\times\left(\frac{\sigma\hbar}{2}\rho\vec{\mu}\right)\right]_{v=0} - \frac{\sigma\hbar}{2c^2}\rho\mathring{\vec{\mu}}\times\left[\frac{d\vec{v}}{d\tau}\right]_{v=0} \quad \text{(C-35)} \\ &= \left[\vec{\nabla}\times\left(\frac{\sigma\hbar}{2}\rho\vec{\mu}\right)\right]_{v=0} + \left[\rho\frac{d}{d\tau}\left(\frac{\vec{v}}{c^2}\times\frac{\sigma\hbar}{2}\mathring{\vec{\mu}}\right)\right]_{v=0} \end{aligned}$$

If we interpret $\frac{\sigma\hbar}{2}\rho\vec{\mu}$ as the particle spin angular momentum density, then the first term on the right side of (C-35) is just the contribution to the linear momentum density arising from the spin angular momentum density. This linear momentum density is completely analogous to the equivalent current density in a magnetized body that is given by the curl of the magnetization.

In order to understand the second term on the right side of (C-35), we first note that a body whose 3-velocity is $\vec{v}$ and whose mass and angular momentum in its rest-frame are respectively $\breve{m}$ and $\frac{\sigma\hbar}{2}\mathring{\vec{\mu}}$ experiences a displacement $\vec{d}$ of its center-of-mass given by the well-known relation:(*)

$$\vec{d} = \frac{\vec{v}\times\left(\frac{\sigma\hbar}{2}\mathring{\vec{\mu}}\right)}{\breve{m}c^2} \tag{C-36}$$

Thus (C-35) may be written

$$\sigma\hbar\,\rho\,\mathring{\vec{\pi}} = \left[\vec{\nabla}\times\left(\frac{\sigma\hbar}{2}\rho\vec{\mu}\right)\right]_{v=0} + \left[\rho\frac{d(\breve{m}\vec{d})}{d\tau}\right]_{v=0} \tag{C-37}$$

The second term on the right side of (C-37) is the momentum density that results from the change with time of the mass moment of a particle.

(*) See, for example, ref. 10, p. 172, eq. 48. (The difference in sign between this equation and (C-36) results merely from a difference in sign in the definition of velocity.)

Since $\tilde{m}$, which is defined in (C-33b), is an invariant, its value in the particle rest-frame equals its value in all frames. Thus

$$\rho\tilde{m} = -\frac{\sigma\hbar}{2c}\left[\partial_k s^{k0}\right]_{v=0} = c^{-2}\left[\vec{\nabla}\cdot\left\{(\rho u^0)\frac{\vec{v}\times(\frac{\sigma\hbar}{2}\mathring{\vec{\mu}})}{c^2}\right\}\right]_{v=0} \qquad \text{(C-38)}$$

where we have made use of (5-2b). Using (C-36) in (C-38), we have

$$\rho\tilde{m} = -\left[\vec{\nabla}\cdot(\rho u^0\breve{m}\vec{d})\right]_{v=0} \qquad \text{(C-39)}$$

Thus $\rho\tilde{m}$ is just the contribution to the mass density that results from the center-of-mass displacement given by (C-36).

Thus, when we impose the constraint (C-26), the vector equation (C-32) can be written in the particle rest-frame as follows:

$$\sigma\hbar\rho\mathring{\omega}^0 = \rho\breve{m}c - c\left[\vec{\nabla}\cdot(\rho u^0\breve{m}\vec{d})\right]_{v=0} \quad \text{for } \cos\epsilon = \sigma \qquad \text{(C-40a)}$$

$$\begin{aligned}\sigma\hbar\rho\mathring{\vec{\omega}} &= \left[\sigma\hbar\rho(\omega^1, \omega^2, \omega^3)\right]_{v=0} \\ &= \left[\vec{\nabla}\times(\tfrac{\sigma\hbar}{2}\rho\vec{\mu})\right]_{v=0} + \left[\rho\frac{d(\breve{m}\vec{d})}{d\tau}\right]_{v=0} + \frac{\rho(\sigma q)}{c}\vec{A} \\ &\qquad \text{for } \cos\epsilon = \sigma\end{aligned} \qquad \text{(C-40b)}$$

This shows us that $\sigma\hbar\rho\omega^j$ is just the rigorously correct expression for the 4-vector canonical momentum density including all the spin effects. Since the spin-dependent contribution to the right side of (C-40a) is very small compared with $\rho \breve{m} c$, the right side of (C-40a) is always positive. Because of the presence of the factor σ on the left, we have the result that the sign of $\overset{\circ}{\omega}{}^{\circ}$ depends on the sign of the particle charge. This is exactly what we concluded from (7-14), which resulted from identifying the direction of rotation of $\overset{\circ}{\vec{k}}$ and $\overset{\circ}{\vec{\lambda}}$ about $\overset{\circ}{\vec{\mu}}$ with the sign of the particle charge. Thus we see that identifying $\cos\epsilon$ with σ, the sign of the particle charge, as we did in (C-26b), has as a consequence the fact that, for solutions of the spinor equations (C-25), the direction of rotation of $\overset{\circ}{\vec{k}}$ and $\overset{\circ}{\vec{\lambda}}$ about $\overset{\circ}{\vec{\mu}}$ (all derived from the spinors by means of (7-27)) is dependent on the sign of the particle charge. This proves the consistency of the two ways of identifying the sign of particle charge - either as $\cos\epsilon$, or as the direction of rotation of $\overset{\circ}{\vec{k}}$ about $\overset{\circ}{\vec{\mu}}$.

We can derive the equation of motion (6-6a) by substituting (C-32) into (7-44). Doing this, we have

$$-\partial^j \Phi = M c u^j + \frac{\sigma q}{c}(A^j + a^j) \quad \text{for } \cos\epsilon = \sigma \qquad \text{(C-41a)}$$

where

$$M = \breve{m} + \tilde{m} \qquad \text{(C-41b)}$$

is to be regarded as the total mass per particle and

$$a^{j} = \frac{\hbar c}{q}(\pi^{j} - \gamma^{j}) = \frac{q}{\alpha}(\pi^{j} - \gamma^{j}) \qquad \text{(C-41c)}$$

where

$$\alpha = \frac{q^{2}}{\hbar c} = \frac{1}{137.04} \qquad \text{(C-41d)}$$

is the fine-structure constant and π^{j} and γ^{j} are defined in (C-33c) and (7-42) respectively. a^{j} is to be regarded as an effective electromagnetic potential to be associated with the magnetic moment of the particle. It is to be noted that, whereas A^{j} is the same in the dynamical equations for each of the two charged fluids, a^{j} in (C-41a) represents the two different potentials $a^{j}_{(+)}$ and $a^{j}_{(-)}$.

We have already seen in Section III that (C-41) suffices to yield a complete dynamical description of each charged fluid for the case of given thermal, gravitational, and electromagnetic fields.

Equation (C-39) gives what might be called a kinetic explanation of the mass increment $\tilde{m}$, based as it is on the center-of-mass displacement produced by the motion of the particle. We shall now derive an electromagnetic explanation. First, we arrive at an expression for $(\partial^j u^k - \partial^k u^j)$ by taking the curl of (C-41a):

$$(\partial^j u^k - \partial^k u^j) = u^j \partial^k \ln M - u^k \partial^j \ln M - \frac{\sigma q}{Mc^2}(F^{jk} + f^{jk}) \quad \text{(C-42a)}$$

where

$$f^{jk} = \partial^j a^k - \partial^k a^j \quad \text{(C-42b)}$$

Now we note that (C-33b) can be rewritten as follows:

$$\tilde{m} = -\frac{\sigma\hbar}{2c\rho} u_k \partial_j (\rho s^{jk}) = \frac{\sigma\hbar}{2c} s^{jk} \partial_j u_k = \frac{\sigma\hbar}{4c} s^{jk} (\partial_j u_k - \partial_k u_j) \quad \text{(C-43)}$$

where we have made use of (C-8a) and the antisymmetry of s^{jk}. Substituting (C-42) into (C-43), we have

$$\tilde{m} = \frac{q\hbar}{4Mc^3} s^{jk}(F_{jk} + f_{jk}) = \frac{q\hbar}{4Mc^3} \mu_j \left[\varepsilon^{jk\ell n} u_k (F_{\ell n} + f_{\ell n})\right] \quad \text{(C-44)}$$

where, in the second step, we have used (C-6). Evaluating the invariant $\tilde{m}c^2$ in the particle rest-frame, we have

$$\tilde{m}c^2 = -\left[\frac{q}{Mc}\left(\frac{\hbar}{2}\overset{\circ}{\vec{\mu}}\right)\cdot\left(\overset{\circ}{\vec{B}} + \overset{\circ}{\vec{b}}\right)\right] \quad \text{(C-45)}$$

where we have used (2-20) and (5-6). Thus $\tilde{m}$ is the mass corresponding to the interaction energy of a magnetic dipole $\frac{q\hbar}{2Mc}\overset{\circ}{\vec{\mu}}$ in a magnetic field $(\overset{\circ}{\vec{B}} + \overset{\circ}{\vec{b}})$. We note that the magnitude of the gyromagnetic ratio is $\frac{q}{Mc}$, which is the same value we obtain from the Dirac Theory, except that in our case the particle mass M includes the thermal, gravitational, and dipole-interaction energy contributions to the mass, as well as the rest mass.

Magnitude of Spin-Dependent Electromagnetic Field

Now let us estimate the magnitudes of a^j and f^{jk}. If we regard the magnitudes of u^j, μ^j, and s^{jk} as being of order unity, and consider only solutions for which these quantities, as well as $\ln \rho$, change significantly only over a distance of order L and in times intervals greater than L/c, then from (C-33c) we have

$$|\pi^j| \sim \frac{1}{L} \tag{C-46}$$

Thus from (7-43), (C-46), and (C-41c), we have

$$|a^j| \sim \frac{q}{\alpha L} \tag{C-47}$$

and from (C-42b), it follows that

$$|f^{jk}| \sim \frac{q}{\alpha L^2} \tag{C-48}$$

Gauge Transformations

Now let us study the effect of a gauge transformation on equation (C-41a). Let $\tilde{A}^j$ and A^j differ only by the gauge transformation generated by the scalar function η, i.e.

$$A^j = \tilde{A}^j + \partial^j \eta \tag{C-49}$$

Substituting this into (C-41a), we have

$$-\partial^j\left(\Phi+\sigma\frac{q}{c}\eta\right)=Mcu^j+\frac{\sigma q}{c}\left(\tilde{A}^j+a^j\right) \tag{C-50}$$

Thus we see that a gauge transformation is equivalent to adding a scalar function to the de Broglie phase function Φ (or Hamilton's Characteristic Function).

If $\tilde{\rho}^\alpha$ and $\tilde{\chi}^\alpha$ are the solutions to the spinor equations (C-25) in which $\tilde{A}^j$ is used to describe the electromagnetic field, whereas ρ^α and χ^α are the solutions resulting when the field is described by A^j, then we find from (C-50), (7-8), and (7-33), or by direct substitution of (C-49) into (C-25), that

$$\tilde{\rho}^\beta=e^{-i\left(\frac{q}{\hbar c}\eta\right)}\rho^\beta=e^{-i\alpha\acute{\eta}}\rho^\beta \tag{C-51a}$$

$$\tilde{\chi}^\beta=e^{+i\left(\frac{q}{\hbar c}\eta\right)}\chi^\beta=e^{+i\alpha\acute{\eta}}\chi^\beta \tag{C-51b}$$

where

$$\acute{\eta}=\frac{\eta}{q} \tag{C-51c}$$

is the dimensionless form of η, and α is the fine-structure constant introduced in (C-41d). Thus we see that a gauge transformation is equivalent to a phase change of the spinors.

From (C-50) it is evident that it is always possible to choose η so that Φ has the form

$$\Phi = -mc^2 t \tag{C-52}$$

Using (7-8), (7-33), and (C-52), we see that, for such a choice of gauge, ρ^{β} and χ^{β} may be written as follows:

$$\rho^{\beta} = e^{i\left(\frac{\sigma mc^2}{\hbar}t\right)}\left[\left(\frac{\rho e^{i\epsilon}}{\sqrt{2}}\right)^{1/2} \hat{\rho}^{\beta}\right] \tag{C-53a}$$

$$\chi^{\beta} = e^{-i\left(\frac{\sigma mc^2}{\hbar}t\right)}\left[\left(\frac{\rho e^{i\epsilon}}{\sqrt{2}}\right)^{1/2} \hat{\chi}^{\beta}\right] \tag{C-53b}$$

The significance of this way of writing the spinors is that it shows that they can always be written in a form such that the fast time dependence is completely contained in a phase factor whose time dependence is given by the de Broglie frequency but which has no spatial dependence. For a magnetogas-dynamical system in which all the observable quantities are time-independent, the square brackets in (C-53) are completely time independent. In such a case ρ^{β} and χ^{β} are irreducible representations of the symmetry operation consisting of a displacement of the origin of the observer's time axis. Thus it would seem that the choice of gauge that corresponds to (C-52), and hence to (C-53), has a fundamental significance that other choices, such as the Lorentz Gauge, do not enjoy.

Error Caused by Variable ϵ

The relations (C-41) assumes that the constraint (C-26) is satisfied. In order to estimate the effect of neglecting this constraint, we substitute (C-32) for arbitrary ϵ into (7-44) and obtain

$$-\partial^j \Phi = \left[(\sigma \cos\epsilon)\breve{m} - \frac{\sigma\hbar}{2c}\mu^k \partial_k \epsilon + \tilde{m}\right] c u^j + \frac{\sigma q}{c}\left(A^j + a^j + \frac{\hbar}{2q}\mu^j \frac{d\epsilon}{d\tau}\right) \tag{C-54}$$

Thus the effect of not holding ϵ constant is to introduce an error $\Delta_\epsilon m$ into the mass and an error $\Delta_\epsilon A^j$ into the 4-potential, where

$$\Delta_\epsilon m = -\breve{m}(1 - \sigma\cos\epsilon) - \frac{\sigma\hbar}{2c}\mu^k \partial_k \epsilon \tag{C-55}$$

$$\Delta_\epsilon A^j = \frac{\hbar}{2q}\mu^j \frac{d\epsilon}{d\tau} = \frac{\hbar c}{2q}\mu^j u^k \partial_k \epsilon \tag{C-56}$$

Using (C-30), we have

$$|1 - \sigma\cos\epsilon| \sim (\lambda\!\!\!^{-}/L)^2 \tag{C-57}$$

$$\left|\frac{\sigma\hbar}{2c}\mu^k \partial_k \epsilon\right| \sim \frac{\hbar}{2c}\left|\frac{\epsilon}{L}\right| \sim \frac{\hbar}{2c}\frac{\lambda\!\!\!^{-}}{L^2} = \frac{m}{2}\left(\frac{\lambda\!\!\!^{-}}{L}\right)^2 \tag{C-58}$$

$$|\Delta_\epsilon A^j| \sim \frac{\hbar c}{2q}\left|\frac{\epsilon}{L}\right| \sim \frac{\hbar c}{2q}\frac{\lambda\!\!\!^{-}}{L^2} = \left(\frac{q}{2\alpha L}\right)\left(\frac{\lambda\!\!\!^{-}}{L}\right) \tag{C-59}$$

For the purposes of an estimate, we may replace $\breve{m}$ in (C-55) with m. Thus from (C-55), (C-57), and (C-58), we have

$$|\Delta_\epsilon m| \sim m(\lambda/L)^2 \tag{C-60}$$

Using (C-47), we can write (C-59) as follows:

$$|\Delta_\epsilon \dot{A}^j| \sim |\dot{a}^j|(\lambda/L) \tag{C-61}$$

which shows immediately that, for all macroscopic dynamical systems, $\Delta_\epsilon \dot{A}^j$ is a fortiori negligible if $\dot{a}^j$ is negligible.

In order to gain an intuitive feeling for the significance of (C-60), let us regard $\Delta_\epsilon m$ as being caused by an error $\Delta_\epsilon h$ in the specific enthalpy. Since the effective mass associated with h is $m(h/c^2)$, we have

$$|\Delta_\epsilon h| \sim c^2(\lambda/L)^2 \tag{C-62}$$

Using (2-2b) and (2-9), $\Delta_\epsilon h$ can in turn be expressed as an equivalent error $\Delta_\epsilon T$ in the temperature:

$$\begin{aligned} |\Delta_\epsilon T| &\sim \frac{m}{k}|\Delta_\epsilon h| \sim \frac{mc^2}{k}(\lambda/L)^2 \\ &\sim \frac{10^{-14}\,\mathrm{cm}^2\,{}^\circ\mathrm{K}}{L^2} \text{ for protons} \\ &\sim \frac{10^{-11}\,\mathrm{cm}^2\,{}^\circ\mathrm{K}}{L^2} \text{ for electrons} \end{aligned} \tag{C-63}$$

which shows immediately that, for systems on a macroscopic scale, $\Delta_\epsilon T$ (and hence $\Delta_\epsilon m$) is completely negligible.

Precession of Particle Spin

Substituting the spinor equations of motion (C-25) into the complex identity (C-18) yields two real vector equations. One of these is (C-32). The other is

$$u^j \partial_j \mu^k = \mu_j \left[(\partial^j u^k - \partial^k u^j) - \varepsilon^{jk\ell n} u_\ell \partial_n \epsilon \right] \qquad \text{(C-64)}$$

where we have made use of (C-27) and (C-28).

As a preliminary to explaining the physical significance of (C-64), we first introduce the 4-vector Ω^j defined as follows:

$$\Omega^j \equiv \tfrac{c}{2}\, \varepsilon^{jk\ell n} u_k \partial_\ell u_n \qquad \text{(C-65)}$$

The physical significance of Ω^j becomes clear when we consider its value in the fluid rest-frame:

$$\overset{\circ}{\Omega}{}^0 = 0 \quad ; \qquad \overset{\circ}{\vec{\Omega}} = (\Omega^1, \Omega^2, \Omega^3)_{v=0} = \tfrac{1}{2} (\vec{\nabla} \times \vec{v})_{v=0} \qquad \text{(C-66)}$$

Thus Ω^j is just the four-dimensional generalization of the local angular velocity associated with the fluid vorticity.

The following identity follows from the definition of Ω^j:

$$(\partial^j u^k - \partial^k u^j) = \tfrac{2}{c}\, \varepsilon^{jk\ell n} u_\ell \Omega_n + (u^j u^\ell \partial_\ell u^k - u^k u^\ell \partial_\ell u^j) \qquad \text{(C-67)}$$

Thus

$$\mu_j(\partial^j u^k - \partial^k u^j) = \frac{2}{c}\mu_j(\varepsilon^{jk\ell n} u_\ell \Omega_n) - \frac{1}{c} u^k \mu_j \frac{du^j}{d\tau} \qquad \text{(C-68)}$$

and (C-64) becomes

$$\frac{d\mu^k}{d\tau} = 2\mu_j \varepsilon^{jk\ell n} u_\ell(\Omega_n - \frac{c}{2}\partial_n\epsilon) - u^k\mu_j\frac{du^j}{d\tau} \qquad \text{(C-69)}$$

In the fluid rest-frame we have

$$\frac{d\mathring{\mu}^0}{d\tau} = \frac{1}{c}\left(\vec{\mu}\cdot\frac{d\vec{v}}{d\tau}\right)_{v=0} \qquad \text{(C-70a)}$$

$$\frac{d\mathring{\vec{\mu}}}{d\tau} = 2\left[(\vec{\Omega} - \frac{c}{2}\vec{\nabla}\epsilon)\times\vec{\mu}\right]_{v=0} \qquad \text{(C-70b)}$$

Equation (C-70a) is just an identity. This is most easily seen by noting that the last term on the right side of (C-69) can be transformed as follows:

$$-u^k\mu_j\frac{du^j}{d\tau} = +u^k u_j\frac{d\mu^j}{d\tau} \qquad \text{(C-71)}$$

Making this transformation in (C-69), (C-70a) becomes the identity

$$\frac{d\mathring{\mu}^0}{d\tau} = \left(u^0 u_j\frac{d\mu^j}{d\tau}\right)_{v=0} = \left(\frac{d\mu}{d\tau}\right)_{v=0} \qquad \text{(C-72)}$$

As a preliminary to interpreting (C-70b), we substitute (C-42) into (C-65) and obtain

$$\Omega^j = -\frac{\sigma q}{4Mc}\varepsilon^{jk\ell n}u_k(F_{\ell n} + f_{\ell n}) \qquad \text{(C-73)}$$

In the fluid rest-frame this becomes

$$\overset{\circ}{\Omega}{}^{\circ} = 0 \; ; \qquad \overset{\circ}{\vec{\Omega}} = -\frac{\sigma q}{2Mc}\left(\overset{\circ}{\vec{B}} + \overset{\circ}{\vec{b}}\right) \tag{C-74}$$

which is just the Larmor Condition we met in (2-44). Thus (C-73) is the relativistic statement of the Larmor Condition.

If we substitute (C-74) into (C-70b), we obtain

$$\frac{d\left(\frac{\sigma\hbar}{2}\overset{\circ}{\vec{\mu}}\right)}{d\tau} = \left(\frac{q\hbar}{2Mc}\overset{\circ}{\vec{\mu}}\right) \times \left(\overset{\circ}{\vec{B}} + \overset{\circ}{\vec{b}}\right) + \frac{\sigma\hbar c}{2}\overset{\circ}{\vec{\mu}} \times \overset{\circ}{\vec{\nabla}}\epsilon \tag{C-75}$$

Neglecting for the moment the second term on the right, which vanishes when we impose the constraint (C-26), we see that this equation describes the precession in its own rest-frame of a particle having spin angular momentum $\frac{\sigma\hbar}{2}\overset{\circ}{\vec{\mu}}$ and a magnetic dipole moment $\frac{q\hbar}{2Mc}\overset{\circ}{\vec{\mu}}$ in a magnetic field $(\overset{\circ}{\vec{B}} + \overset{\circ}{\vec{b}})$. These values correspond to the gyromagnetic ratio $\frac{\sigma q}{Mc}$.

The second term on the right corresponds to an additional precession whose angular velocity is $-c\vec{\nabla}\epsilon$. With the same basic assumptions used in making previous estimates, we find

$$\frac{\left|\frac{\sigma\hbar c}{2}\overset{\circ}{\vec{\mu}} \times \overset{\circ}{\vec{\nabla}}\epsilon\right|}{\left|\frac{q\hbar}{2Mc}\overset{\circ}{\vec{\mu}} \times \left(\overset{\circ}{\vec{B}} + \overset{\circ}{\vec{b}}\right)\right|} \sim \frac{\hbar c}{q\left|\overset{\circ}{\vec{B}} + \overset{\circ}{\vec{b}}\right| L^2} \tag{C-76}$$

In the case of a system of laboratory dimensions, we can take $L \sim 1$ cm and $|\overset{\circ}{\vec{B}}| \sim 1$ gauss. We have already seen that for such a case that $|\vec{\mathring{b}}| \ll |\vec{\mathring{B}}|$, and so $\vec{\mathring{b}}$ can be neglected. Thus we find that the ratio (C-76) of the precession velocities is of order 10^{-7}. For larger L or $\vec{\mathring{B}}$, the ratio would be even smaller. Thus, for macroscopic systems, the spurious precession resulting from not holding ϵ constant (i.e. not imposing the constraint (C-26)) is completely negligible compared with the real precession given by the first term on the right side of (C-75).

Finally we note that, substituting the spinor equations (C-25) into (C-19), we obtain from the imaginary antisymmetric part of this equation

$$\begin{aligned} u^{\ell}\partial_{\ell} \dot{s}^{jk} = {} & s^{j}{}_{\ell}\left[(\partial^{\ell}u^{k} - \partial^{k}u^{\ell}) - \varepsilon^{\ell k p q} u_{p}\partial_{q}\epsilon\right] \\ & - s^{k}{}_{\ell}\left[(\partial^{\ell}u^{j} - \partial^{j}u^{\ell}) - \varepsilon^{\ell j p q} u_{p}\partial_{q}\epsilon\right] \end{aligned} \qquad \text{(C-77)}$$

where we have made use of (C-28) and (C-32). (C-77) is equivalent to (C-64), except that the spin has been represented by the tensor $\dot{s}^{jk}$ instead of the vector $\dot{\mu}^{j}$. The 3-vector equation obtained by writing the space-like components of (C-77) in the particle rest-frame is just (C-70b), which has already been discussed.

<u>Stress-Energy Tensor</u>

Substituting the spinor equations of motion (C-25) into (C-21), the imaginary part of the resulting equation yields

$$\begin{aligned} \not{s}\left[\partial_{\bar{\alpha}\beta}(\xi^{\bar{\alpha}}\partial_{k}\xi^{\beta} - \chi^{\bar{\alpha}}\partial_{k}\chi^{\beta})\right] &= \partial^{j} T_{jk} \qquad \text{(C-78)} \\ &= \frac{\rho c}{\hbar} \cos\epsilon\, \partial_{k}\check{m} + \frac{\rho q}{\hbar c} u^{\ell}\partial_{k} A_{\ell} \end{aligned}$$

where $\mathfrak{T}_{jk}$ is defined in (C-23). As a preliminary to explaining this relation, let us define a 4-vector p^j as follows:

$$p^j = \sigma\hbar\omega^j - \frac{\sigma q}{c} A^j \tag{C-79}$$

Since $\sigma\hbar c\omega^j$ has been interpreted as the total (or canonical) particle momentum, including spin effects and the contribution of the external electromagnetic field, p^j is just the inertial part of the particle momentum, including spin effects. From (C-32) we have

$$p^j = (\sigma\cos\epsilon)\breve{m}cu^j - \frac{\sigma\hbar}{2\rho}\partial_k(\rho s^{kj}) - \frac{\sigma\hbar}{2}\hat{s}^{jk}\partial_k\epsilon \tag{C-80}$$

Using (C-33), this can also be written

$$p^j = \left[(\sigma\cos\epsilon)\breve{m} + \tilde{m}\right]cu^j + \sigma\hbar\pi^j - \frac{\sigma\hbar}{2}\hat{s}^{jk}\partial_k\epsilon \tag{C-81}$$

Now let us define the tensor T_{jk} as follows:

$$\begin{aligned} T_{jk} &\equiv \sigma\hbar c\,\mathfrak{T}_{jk} - \sigma q\rho u_j A_k \\ &= c\rho\left[u_j p_k + \frac{\sigma\hbar}{2}w_j\partial_k\epsilon - \frac{\sigma\hbar}{2}s_{j\ell}\partial_k u^\ell\right] \end{aligned} \tag{C-82}$$

From (C-82), (C-21), and (C-23) we have

$$T^{jk} = \sigma\hbar c\,\mathcal{I}\left[\sigma^j_{\bar\alpha\beta}(\xi^{\bar\alpha}\partial^k\xi^\beta - \chi^{\bar\alpha}\partial^k\chi^\beta)\right] - \sigma q\,\sigma^j_{\bar\alpha\beta}(\xi^{\bar\alpha}\xi^\beta + \chi^{\bar\alpha}\chi^\beta)A^k \tag{C-83}$$

which expresses T^{jk} directly in terms of the spinors ξ^α and χ^α.

Substituting (C-82) into (C-78) and using (C-28), we have

$$\partial^j T_{jk} = \rho(\sigma \cos \epsilon)\, \partial_k(\breve{m} c^2) + \sigma q \rho\, (\partial_k A_\ell - \partial_\ell A_k) u^\ell \qquad \text{(C-84)}$$

For the case in which condition (C-26) is satisfied, (C-84) becomes

$$\begin{aligned} \partial^j T_{jk} &= \rho\, \partial_k(\breve{m} c^2) + \sigma q \rho F_{k\ell} u^\ell \\ &= \mathfrak{k}_k \end{aligned} \qquad \text{for } \cos \epsilon = \sigma \qquad \text{(C-85)}$$

where $\mathfrak{k}_k$ is the force density defined in (6-16).

We note that (C-85) is identical with (6-15) except that the contraction on the left side of (C-85) involves the first index, instead of the second as in (6-15). We shall now show that $\partial^j T_{jk} = \partial^j T_{kj}$ (which is not obvious, since T_{jk} is not symmetric).

Substituting the spinor equations of motion into the identity (C-19), we have from the real antisymmetric part of the resulting equation the following relation:

$$\tfrac{1}{2}\left[\partial^j(\rho\mu^k) - \partial^k(\rho\mu^j)\right] - \varepsilon^{jk\ell n}\, \mathfrak{J}_{\ell n} = -\frac{q\rho}{\hbar c}\, \varepsilon^{jk\ell n} u_\ell A_n \qquad \text{(C-86)}$$

Contracting this with ε_{pqjk} and using (C-82), we arrive at the following:

$$\frac{1}{c}(T_{jk} - T_{kj}) = -\frac{\sigma\hbar}{2}\, \varepsilon_{jk\ell n}\, \partial^\ell(\rho\mu^n) \qquad \text{(C-87)}$$

Because the divergence of the right side vanishes, we have the desired result:

$$\partial^j T_{jk} = \partial^j T_{kj} \tag{C-88}$$

Thus (C-85) is identical with (6-15), and we are justified in interpreting T_{jk} as the stress-energy tensor for the fluid.

Substituting (C-80) into (C-82), we have for the case $\cos\epsilon = \sigma$

$$T^{jk} = \rho \breve{m} c^2 u^j u^k - \frac{\sigma\hbar c}{2} u^j \partial_\ell (\rho s^{\ell k}) - \frac{\sigma\hbar c}{2} \rho s^{j\ell} \partial_k u_\ell \qquad \text{for } \cos\epsilon = \sigma \tag{C-89}$$

Comparing with (6-11), we have

$$t^{jk} = -\frac{\sigma\hbar c}{2}\left[u^j \partial_\ell (\rho s^{\ell k}) + \rho s^{j\ell} \partial_k u_\ell\right] \qquad \text{for } \cos\epsilon = \sigma \tag{C-90}$$

where t^{jk} is the spin-dependent part of T^{jk}.

As discussed in Section V, the fluid energy density is given by T^{oo}; the fluid energy flux density $\mathcal{U}^j_{f\ell}$ (designated by $\mathcal{U}^j_{(\pm)}$ in Section V) is given by cT^{oj}; and the fluid momentum density $\mathcal{G}^j_{f\ell}$ (designated by $\mathcal{G}^j_{(\pm)}$ in Section V) is given by $\frac{1}{c}T^{jo}$. It can be shown that in the fluid rest-frame these quantities assume the following form:

$$\mathring{T}^{oo} = \rho(\breve{m} + \hat{m})c^2 \qquad \text{for } \cos\epsilon = \sigma \tag{C-91}$$

$$\overset{\circ}{\vec{\mathcal{U}}}_{\mu} = c^2 \rho \left[\frac{d(\breve{m}\vec{d})}{d\tau}\right]_{v=0} + c^2 \left[\vec{\nabla} \times \rho\left(\frac{\sigma\hbar}{2}\vec{\mu}\right)\right]_{v=0} \quad \text{for } \cos\epsilon = \sigma \tag{C-92}$$

$$\overset{\circ}{\vec{\mathcal{G}}}_{\mu} = \rho\left[\frac{d(\breve{m}\vec{d})}{d\tau}\right]_{v=0} \quad \text{for } \cos\epsilon = \sigma \tag{C-93}$$

where $\vec{d}$ is defined in (C-36). These expressions make the role of particle spin intuitively evident. The first term on the right side of (C-92) is the energy flux arising from the change in displacement of the center-of-mass of an accelerating particle having spin. $\overset{\circ}{\vec{\mathcal{G}}}_{\mu}$ is just the corresponding momentum density. The second term on the right side of (C-92) is the energy flux associated with the linear momentum density arising from the spin angular momentum. If this term (divided by c^2) were to appear in $\overset{\circ}{\vec{\mathcal{G}}}_{\mu}$ as well, it would, when we came to calculate the angular momentum density $\vec{r} \times \vec{\mathcal{G}}_{\mu}$ associated with T^{jk}, give rise to an angular momentum contribution that had its origin in the particle spin. The fact that this term does not appear in $\overset{\circ}{\vec{\mathcal{G}}}_{\mu}$ is consistent with the fact that we treat the spin angular momentum density $\vec{\mathcal{L}}$ as a separate contribution that is not included in $\vec{r} \times \vec{\mathcal{G}}_{\mu}$.

<u>Angular Momentum</u>

From (C-85) and (C-88), we obtain the following:

$$(T^{jk} - T^{kj}) = -\partial_{\ell}(x^j T^{k\ell} - x^k T^{j\ell}) + (x^j \mathfrak{k}^k - x^k \mathfrak{k}^j) \tag{C-94}$$

Substituting this into (C-87), we have

$$\partial_\ell M^{jk\ell} = \frac{1}{c}(x^j \mathscr{k}^k - x^k \mathscr{k}^j) \tag{C-95a}$$

where

$$M^{jk\ell} = \mathscr{s}^{jk\ell} + \frac{1}{c}(x^j T^{k\ell} - x^k T^{j\ell}) \tag{C-95b}$$

and

$$\mathscr{s}^{jk\ell} = -\frac{\sigma\hbar}{2}\varepsilon^{jk\ell n}(\rho\mu_n) \tag{C-95c}$$

As shown in Section V, (C-95a) and (C-95b) constitute the differential form of the statement of conservation of angular momentum, whose integral form is

$$\frac{d}{dt}\int_{V_3}(\vec{\mathscr{s}} + \vec{r}\times\vec{\mathscr{g}}_{\mu\ell})\,dV_3 = \int_{V_3}(\vec{r}\times\vec{\mathscr{k}})\,dV_3 \tag{C-96a}$$

where from (C-95c) we have

$$\vec{\mathscr{s}} \equiv (\mathscr{s}^{230}, \mathscr{s}^{310}, \mathscr{s}^{120}) = \frac{\sigma\hbar}{2}\rho\vec{\mu} = (\rho u^0)\left(\frac{\sigma\hbar}{2}\frac{\vec{\mu}}{u^0}\right) \tag{C-96b}$$

The integrand on the left in (C-96a) is the total angular momentum density, and the integrand on the right is the torque density. Since $\vec{\mathscr{s}}$ is the spin angular momentum density, and (ρu^0) is the particle density in the observer's reference frame, we see from (C-96b) that $\frac{\sigma\hbar}{2}\frac{\vec{\mu}}{u^0}$ is to be interpreted as the spin angular momentum of a single particle as seen in the observer's frame.

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